Technical Appendix for: "A Competitive Model of the Informal Sector"

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1. Proofs for section 4

1.1. Proof of lemma 1

We will show this result for a large class of constraints. Consider the following problem:

$$V(a, z; \eta, \tau) = \max_{s \leq a, d \geq 0} (1 - \tau)\Pi(s + d, z)$$

$$s.t.\ \Phi(s, d, z; \eta, \tau) \geq 0,$$

where:

Assumption 1 Φ is twice continuously differentiable and concave. Furthermore, $\Phi_1 > 0$, $\Phi_3 > 0$, and $\Phi_{12} \leq 0$.

Clearly, the enforcement technology we specify in the text leads to an incentive compatibility constraint that satisfies assumption 1. The concavity of Φ enables us to use standard tools. The main element of the assumption is that Φ_{12} be negative. Because s and d enter additively

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the first argument of a concave function (Π), many natural specification of the enforcement technology satisfy that requirement.

Define $a^*(z;\eta,\tau) = inf\{a : s(a,z;\eta,\tau) + d(a,z;\eta,\tau) = k^*(z)\}$, where s and d are the optimal solutions to the general problem stated above. This is the asset level below which a manager is borrowing-constrained. We will show that this level of assets satisfies each item of lemma 1. Part (iv) of the lemma is obvious from the definition of $a^*(z;\eta,\tau)$. Note also that $a^*(z;\eta,\tau) \leq k^*(z)$. If $a^*(z;\eta,\tau) = 0$, the lemma holds trivially. So assume that $a > 0$ exists such that $a < a^*(z;\eta,\tau)$. Necessary and sufficient conditions for optimization are:

$$(1-\tau)\Pi_1(s+d,z) + \lambda\Phi_2(s,d,z;\eta,\tau) = 0, \quad (1)$$

$$\lambda\Phi(s,d,z;\eta,\tau) = 0, \quad (2)$$

$$(1-\tau)\Pi_1(s+d,z) + \lambda\Phi_1(s,d,z;\eta,\tau) - \nu = 0, \quad (3)$$

$$\nu(a-s) = 0, \quad (4)$$

where $\lambda \geq 0$ is the Lagrange multiplier associated with the incentive compatibility constraint, and $\nu \geq 0$ is the multiplier associated with the constraint $s \leq a$.

To show part (ii) of the lemma note that since $a < a^*(z;\eta,\tau)$, the manager is borrowing-constrained, hence $\Pi_1 > 0$. From (1) this means $\Phi_2 < 0$. Then note that (1) and (3) imply $\lambda(\Phi_1 - \Phi_2) = \nu$. This in turn means that $\nu > 0$, which, from (4), yields the result.

To show part (iii), note that $\Phi(\cdot,d,z;\eta,\tau) = 0$ in a neighborhood of a. Differentiating w.r.t. a yields $\Phi_1 + \Phi_2\frac{\partial d}{\partial a} = 0$, hence

$$\frac{\partial d}{\partial a} = -\frac{\Phi_1}{\Phi_2} > 0.$$

Differentiating w.r.t. a once more gives

$$\frac{\partial^2 d}{\partial a^2} = -\frac{\Phi_2\left(\Phi_{11} + \Phi_{12}\frac{\partial d}{\partial a}\right) - \Phi_1\left(\Phi_{21} + \Phi_{22}\frac{\partial d}{\partial a}\right)}{\Phi_2^2} < 0,$$

as needed for the third item of the lemma.

Finally, to show part (i), first use the envelope theorem to obtain

$$V_1(a, z; \eta, \tau) = \nu > 0.$$

To show strict concavity, note that $V_{11}(a, z; \eta, \tau) = \frac{\partial \nu}{\partial a}$. But, from (3),

$$\frac{\partial \nu}{\partial a} = (1-\tau)\Pi_{11}\left(1 + \frac{\partial d}{\partial a}\right) + \frac{\partial \lambda}{\partial a}\Phi_1 + \lambda\left(\Phi_{11} + \Phi_{12}\frac{\partial d}{\partial a}\right).$$

By (1),

$$\frac{\partial \lambda}{\partial a} = -(1-\tau)\frac{\Pi_{11}\left(1 + \frac{\partial d}{\partial a}\right)\Phi_2 - \left(\Phi_{21} + \Phi_{22}\frac{\partial d}{\partial a}\right)\Pi_1}{\Phi_2^2} < 0$$

Together with assumption 1 this implies $\frac{\partial \nu}{\partial a} < 0$, and completes the proof.

1.2. Proof of lemma 2

Consider a manager who is borrowing-constrained. Differentiating the incentive compatibility constraint w.r.t z yields $\Phi_3 + \Phi_2 \frac{\partial d}{\partial z} = 0$. Thus, $\frac{\partial d}{\partial z} = -\frac{\Phi_3}{\Phi_2} > 0$, as needed.

1.3. Proof of lemma 3

The first item of the lemma is obvious, so fix p and assume that z is large enough that an agent of type (p, z) chooses to become a manager. Clearly, those agents devote no time to education in the first period. Let $W_I(z) = \max_{a\in[0,w_u)} \log(w_u - a) + \beta \log\left[V(a, z; 0, 0) + a(1+r)\right]$, while $W_F(z) = \max_{a\in[0,w_u)} \log(w_u - a) + \beta \log\left[V(a, z; \eta, \tau) + a(1+r)\right]$. That is, $W_F(z)$ is the maximum lifetime utility that an agent of ability z can obtain if she becomes a formal manager, while $W_I(z)$ is the maximum lifetime utility that a manager of ability z can obtain if she opts for the informal sector. A manager of ability z becomes choose to operate formally when $W_F(z) > W_I(z)$. We will show that whenever $W_F(z) = W_I(z)$, $\frac{\partial}{\partial z}W_F(z) > \frac{\partial}{\partial z}W_I(z)$ for all z, which means that $W_F(z)$ can only intersect $W_I(z)$ from below. Since W_F and W_I are continuous, this implies that they intersect at most once.

So let z be such that, $W_F(z) = W_I(z)$ and let a_I and a_F be the corresponding optimal saving decisions. First note that solving out for l_u implies that $\Pi(k,z) = z^{\frac{1}{1-\theta}}Ck^{\gamma} - k(1 + r + w_s)$ where C is a constant that is independent of z, and $\gamma \equiv \frac{\alpha}{1-\theta} < 1$. Then,

$$\frac{\partial}{\partial z}W_I(z) = \frac{\frac{\beta}{1-\theta}z^{\frac{\theta}{1-\theta}}Ca_I^{\gamma}}{z^{\frac{1}{1-\theta}}Ca_I^{\gamma} - a_I w_s}. \tag{5}$$

Here we are using the fact that because $\beta(1+r) \leq 1$, managers only save in order to self-finance production. Turning to formal managers and letting d stand for $d(a_F, z; \eta, \tau)$, we must have:

$$\frac{\partial}{\partial z}W_F(z) \geq \frac{\frac{\beta(1-\tau)}{1-\theta}z^{\frac{\theta}{1-\theta}}C(a_F+d)^{\gamma}}{(1-\tau)[z^{\frac{1}{1-\theta}}C(a_F+d)^{\gamma} - (a_F+d)(w_s+1+r)] + a_F(1+r)},$$

where the inequality follows from the fact that we are ignoring the positive effect of z on d. To proceed, it will prove convenient to establish that $(1-\tau)d \geq \tau a_F$. To that end, note that we must have

$$\begin{aligned} (1-\tau)\Pi(a_F+d, z) + a_F(1+r) &\geq \Pi(a_F, z) + a_F(1+r) \\ \Longleftrightarrow (1-\tau)(\Pi(a_F+d, z) - \Pi(a_F, z)) &\geq \tau\Pi(a_F, z). \end{aligned}$$

Otherwise, a manager of type z could generate strictly more income in the informal sector, contradicting the premise that $W_F(z) = W_I(z)$. But, since Π is concave in its first argument, $(1-\tau)(\Pi(a_F+d,z) - \Pi(a_F,z)) \leq (1-\tau)d\Pi_1(a_F,z)$ while $\tau\Pi(a_F,z) \geq \tau a_F\Pi_1(a_F,z)$, so we must have $(1-\tau)d \geq \tau a_F$ as claimed. In turn, this implies that

$$\frac{\partial}{\partial z}W_F(z) \geq \frac{\frac{\beta}{1-\theta} z^{\frac{\theta}{1-\theta}} C(a_F+d)^{\gamma}}{z^{\frac{1}{1-\theta}} C(a_F+d)^{\gamma} - (a_F+d)w_s}. \tag{6}$$

Comparing expressions (5) and (6) reveals that $\frac{\partial}{\partial z}W_F(z) > \frac{\partial}{\partial z}W_I(z)$ provided $a_F + d > a_I$. Indeed, $\frac{k}{Ck^{\gamma}}$ rises with k since $\gamma < 1$.

To establish that $a_F + d > a_I$, note that the first-order condition for a_I is:

$$\frac{a_I}{w_u - a_I} = \beta \left[\frac{\gamma z^{\frac{1}{1-\theta}} C a_I^{\gamma-1} - w_s}{z^{\frac{1}{1-\theta}} C a_I^{\gamma-1} - w_s} \right],$$

while optimality for a_F implies

$$\frac{a_F+d}{w_u - a_F} > \beta \left[\frac{\gamma z^{\frac{1}{1-\theta}} C(a_F+d)^{\gamma-1} - w_s}{z^{\frac{1}{1-\theta}} C(a_F+d)^{\gamma-1} - w_s} \right].$$

The inequality follows from the facts that we are dropping term $\tau(1+r) > 0$ from the numerator of the right-hand side of the first order condition for a_F, that we are dropping term $(\tau a_F - (1-\tau)d)(1+r) \leq 0$ from the denominator, and that we are ignoring the positive effect of a on d. Assume now, by way of contradiction, that $a_F + d \leq a_I$. Then we must have

$$\frac{\gamma z^{\frac{1}{1-\theta}} C a_I^{\gamma-1} - w_s}{z^{\frac{1}{1-\theta}} C a_I^{\gamma-1} - w_s} > \frac{\gamma z^{\frac{1}{1-\theta}} C(a_F+d)^{\gamma-1} - w_s}{z^{\frac{1}{1-\theta}} C(a_F+d)^{\gamma-1} - w_s}.$$

But because $\gamma < 1$ this implies $a_F + d > a_I$, the contradiction we sought. This implies that $\frac{\partial}{\partial z}W_F(z) > \frac{\partial}{\partial z}W_I(z)$ as needed. Optimal sector choice is thus fully described by a managerial threshold z^*.

We now need to show that formal managers use more capital than informal managers. We have established that this is the case for $z = z^*$. For managers of other type, let $a_I(z)$ be the optimal saving choice manager of type z would make if they were constrained to operate in the informal sector and $a_F(z)$ be the similar choice when they have to operate in the formal sector, while $d(z)$ is the capital they borrow. The first order condition for $a_I(z)$ implies that a_I rises with z. On the other hand, the same argument as above implies that $a_F(z) + d(z) \geq a_I(z)$ for all $z \geq z^*$. It now follows that for all z_1 and z_2 such that $z_1 < z^* < z_2$,

$$a_I(z_1) < a_I(z_2) < a_F(z_2) + d(z_2),$$

which implies that formal managers operate with more capital than informal managers, and completes the proof.

1.4. Proof of lemma 4

A young agent who anticipates that he will become a manager in the second period devotes no time to education. Suppose, by way of contradiction, that one can find an informal manager of higher managerial talent than a formal manager. Then there exists $z_1 > z_2$ such that $W_F(z_1) \geq W_I(z_1)$ and $W_F(z_1) \leq W_I(z_1)$. Since both functions are continuous and differentiable, this implies that z exists such that $W_F(z) = W_I(z)$ and $\frac{\partial}{\partial z} W_F(z) \leq \frac{\partial}{\partial z} W_I(z)$, a possibility which we ruled out in the proof of lemma 3.

2. Proofs for section 5

2.1. Proof of proposition 5

Denote by $\Phi^{w_u,w_s,p,z}$ the set of measures ϕ with finite support on $[0,1] \times \mathbb{R}^2$ such that $\phi(e, l^u, l^s) > 0$ implies that e, l^u, l^s is an optimal policy for agent type (p, z) given wage rates $w_u, w_s > 0$. Define ED as the mapping which associates with any pair $w_u > 0$ and $w_s > 0$

the set of possible excess demands for unskilled and skilled labor, i.e.:

$$ED(w_u, w_s) = \Big\{\Big(\int [l_u - (1-e)]d\phi^{p,z}d\mu - \int \phi^{p,z}(0,0,0)d\mu, \int [l_s - he^p]d\phi^{p,z}d\mu\Big) : \phi^{p,z} \in \Phi^{w_u,w_s,p,z} \quad \forall (p,z) \in [0,1]^2\Big\}.$$

Since the price of capital and skilled labor is bounded below by $(1+r)$, a value $\bar{w}_u > 0$ exists such that the excess demand for unskilled labor is strictly negative when $w_u \geq \frac{\bar{w}_u}{2}$. Indeed, for w_u large enough no agent chooses to become a manager while given the preferences we have assumed, all agents supply positive amounts of unskilled labor in the first period of their life when $w_u > 0$. There must then exist $\bar{w}_s$ such that:

$$w_s \geq \frac{\bar{w}_s}{2} \Longrightarrow [(x_u, x_s) \in ED(w_s, w_u) \Longrightarrow x_u > 0 \text{ or } x_s < 0] \tag{7}$$

Indeed, if $\frac{w_s}{w_u} \to \infty$, the ratio of the demand for skilled labor to the demand for unskilled labor goes to zero. For every $\epsilon > 0$ define the compact set $A_\epsilon = \{w_u, w_s : \epsilon \leq w_u \leq \bar{w}_u \text{ and } 0 \leq w_s \leq \bar{w}_s\}$. Note that $ED(A_\epsilon)$ is bounded since $w_u \geq \epsilon$ and the price of capital (hence the effective price of skilled labor) is bounded below by $1+r$. Now for all $n \in \mathbb{N}$ define the mapping $T^{\frac{1}{n}} : ED(A_{\frac{1}{n}}) \longrightarrow A_{\frac{1}{n}}$ by

$$T^{\frac{1}{n}}(x_u, x_s) = \arg \max_{(w_u, w_s) \in A_{\frac{1}{n}}} w_u x_u + w_s x_s$$

Intuitively, $T^{\frac{1}{n}}$ raises wages when excess demands are positive and lowers them otherwise. For each agent type, the theorem of the maximum implies that the set of optimal policies is non-empty and upper hemi-continuous. Since different agents of a given type may choose different policies, excess demands are the finite sums of non-empty, upper hemi-continuous and *convex-valued* correspondences on $A_{\frac{1}{n}}$. Standard arguments (see e.g. Hildenbrand and Kirman, 1988, p278) imply that for all $n \in \mathbb{N}$, $T^{\frac{1}{n}}$ is non-empty, upper hemi-continuous and convex-valued. Therefore, by Kakutani's fixed point theorem, $T^{\frac{1}{n}} \times ED$ has a fixed point in $A_{\frac{1}{n}} \times ED(A_{\frac{1}{n}})$ for all $n \in \mathbb{N}$. Let $(w_u^n, w_s^n, x_u^n, x_s^n)$ be a corresponding sequence of fixed points. Since A_0 is compact, assume without loss of generality that (w_u^n, w_s^n) converges to (w_u^*, w_s^*) $\in A_0$. We will argue that (w_u^*, w_s^*) is in the interior of A_0. Note first that $w_u^* = \bar{w}_u$ would imply that $x_u^n < 0$ eventually, hence (by the definition of $T^{\frac{1}{n}}$), $w_u^n = \frac{1}{n}$ eventually, which cannot be if $w_u^* = \bar{w}_u$. If $w_u^* = 0$, $x_u^n > 0$ eventually. That is because $\lim_{k \mapsto 0} \frac{\Pi(k,z)}{k} = +\infty$, which implies that $d(0, z; \eta, \tau) > \underline{d} > 0$ for all $z \in [0, 1]$ and for any pair of wage rates $(w_u, w_s) \leq (\bar{w}_u, \bar{w}_s)$. So the demand for unskilled labor grows without bound when $w_u \to 0$. Hence (by the definition of $T^{\frac{1}{n}}$), $w_u^n = \bar{w}_u$, eventually, a contradiction of the premise that $w_u^* = 0$. Since we have now established that $w_u^* > 0$, (x_u^n, x_s^n) is a bounded sequence, and we may now assume without loss of generality that it converges to (x_u^*, x_s^*).

Now, if $w_s^* = \bar{w}_s$, then, eventually, either $x_s^n < 0$ in which case $w_s^n = 0$, or $x_u^n > 0$ in which case $w_u^n = \bar{w}_u$. But $w_s^* = \bar{w}_s$ implies that $w_s^n = 0$ cannot happen infinitely often, so we must have $w_u^* = \bar{w}_u$, a possibility which we ruled out above. Finally, we need to rule out the possibility that $w_s^* = 0$. If $w_s^* = 0$, the supply of skilled workers is eventually zero since $w_u^* > 0$. If the demand for unskilled labor is positive, the demand for skilled labor must also be positive. So, eventually, we must have either $x_u^n < 0$ or $x_s^n > 0$. This implies that $w_s^n = \bar{w}_s$ infinitely often (which cannot be if $w_s^* = 0$) or $w_u^* = 0$ infinitely often (which cannot be since $w_u^* > 0$.) So (w_u^*, w_s^*) is in the interior of A_0, as claimed. This implies that (w_u^n, w_s^n) is in the interior of $A_{\frac{1}{n}}$ for n large enough, which given the definition of $T^{\frac{1}{n}}$ implies that $(x_u^n, x_s^n) = (0, 0)$ eventually, and completes the proof.

2.2. *Proof of remark 6*

The informal share of employment in steady state can be made as large as desired by raising the tax rate or lowering the default cost parameter since that share is 1 when $\tau = 1$, or when $\eta = 0$ but $\tau > 0$.

2.3. *Proof of proposition 7*

Consider steady states where both sectors coexist. (For other steady states, the result holds vacuously.) The first item of the proposition follows directly from lemma 4. From lemma 3, we also know that formal managers work with more capital. Since demand for unskilled labor rises with both z and k, formal managers must then employ more unskilled workers than informal managers. As for skilled employment, denote the average units of skilled labor offered by each skilled worker by

$$E(ph(e)) = \frac{\int_{e>0} ph(e)\phi^{p,z}(e, l_u, l_s) d\mu(p, z)}{\int_{e>0} \phi^{p,z}(e, l_u, l_s) d\mu(p, z)}.$$

On average, managers with capital k employ $\frac{k}{E(ph(e))}$ skilled workers. Since formal managers operate with more capital, they must therefore operate with more skilled workers. This establishes the second item of the proposition.

To obtain the last two items, consider any formal manager and any informal manager. Let k^F, l_u^F and y^F denote respectively the formal manager's demand for capital, demand for unskilled labor and the corresponding output while k^I, l_u^I and y^I are the corresponding quantities for the informal manager. First order conditions for profit maximization imply that for $j \in I, F$,

$$\theta y^j = w_u l_u^j.$$

In turn, this implies that the formal manager operates at a higher capital to unskilled labor ratio if and only if $\frac{y^F}{k^F} < \frac{y^I}{k^I}$. If the formal manager is unconstrained, this condition holds trivially as $\frac{y^F}{k^F}$ is then at its lowest possible value. Assume then that the formal manager is constrained. We will derive an upper bound for $\frac{y^F}{k^F}$ in that case. To that end, assume that the formal manager has no assets (if they have positive assets, $\frac{y^F}{k^F}$ becomes even lower.) The incentive compatibility constraint then implies:

$$(1-\tau)[(1-\theta)y^F - k^F(1+r+w_s)] = (1-\eta)(1-\tau)[(1-\theta)y^F - k^F w_s]$$
$$\iff \frac{y^F}{k^F} = \frac{w_s}{1-\theta} + \frac{1+r}{\eta(1-\theta)}. \tag{8}$$

We now need a lower bound on $\frac{y^I}{k^I}$. The first order condition for optimal saving by the informal manager yields

$$\frac{1}{w_u - k^I} = \beta\left[\frac{\alpha\frac{y^I}{k^I} - w_s}{(1-\theta)y_I - k^I w_s}\right]. \tag{9}$$

Furthermore, informal managers must be better off using k^I in production than if they chose to deposit it with the intermediary and become unskilled workers. This yields,

$$(1-\theta)y^I - k^I w_s \geq w_u + k^I(1+r)$$
$$\iff \frac{y^I}{k^I} \geq \frac{w_s}{1-\theta} + \frac{1+r}{(1-\theta)} + \frac{w_u}{(1-\theta)k^I}. \tag{10}$$

Comparing (10) and (8) shows that $\frac{y^I}{k^I} > \frac{y^F}{k^F}$ if $\frac{w_u}{k^I} > (1+r)\left(\frac{1}{\eta} - 1\right)$. Assume then that the opposite equality holds. Since $(1-\theta)y^I - k^I w_s \geq w_u + k^I(1+r)$, (9) implies

$$\frac{k^I(1+r) + w_u}{w_u - k^I} \leq \beta\left(\alpha\frac{y^I}{k^I} - w_s\right).$$

In turn $\frac{w_u}{k^I} \leq (1+r)\left(\frac{1}{\eta} - 1\right)$ implies

$$\frac{y^I}{k^I} \geq \frac{w_s}{\alpha} + \frac{1}{\alpha\beta}\left[\frac{1}{\frac{(1+r)(1-\eta)}{\eta} - 1}\right]\frac{1+r}{\eta}.$$

Comparing this last inequality to (8) and using the fact that $\alpha < 1-\theta$ shows that $\frac{y^F}{k^F} < \frac{y^I}{k^I}$

provided

$$
\begin{aligned}
\alpha\beta\left[\frac{(1+r)(1-\eta)}{\eta}-1\right] &< 1-\theta \\
\Longleftrightarrow \frac{1-\eta}{\eta} &< \frac{1-\theta}{\alpha\beta(1+r)}+\frac{1}{1+r}=\frac{1-\theta+\alpha\beta}{\alpha\beta(1+r)}.
\end{aligned}
$$

Under that condition, formal managers operate at a higher capital to unskilled ratio that informal managers, hence at a higher skilled to unskilled ratio, hence at a higher capital to employment ratio. This completes the proof.

3. Characterization of education policy functions

In equilibrium, agents who eventually become managers devote no time to education. As for other agents, standard manipulations of first order conditions imply that the time devoted to education rises with the education type. In fact, a threshold exists such that agents whose education type exceeds that threshold, and those agents only, become skilled workers in the second period. The following lemma records those results. We omit the proof for conciseness.

Lemma 8 *In steady state, for all* $z \geq 0$,

1. $e(p,z)$ *rises with* $p \in [0,1]$, *while* $a(p,z)$ *decreases with* p;
2. *There exists* $\underline{p} \in [0,1]$ *such that* $p > \underline{p} \Longrightarrow e(p) > 0$ *and* $p < \underline{p} \Longrightarrow e(p) = 0$.

References

Hildenbrand, W. and Kirman, A. P., 1988. Equilibrium Analysis. North Holland, Amsterdam, The Netherlands.