CAPM: a formal proof

Take asset i in our financial economy with risky payoff X_i. What should be its equilibrium price q_i? Equivalently, what should be its average return $\bar{r}_i = \frac{E(X_i)}{q_i} - 1$? Because investors are risk-averse the price should depend on the risk the security is adding to the market portfolio. The greater the risk it adds to that portfolio, the lower the price ought to be. The higher, in other words, the return investors will demand in equilibrium to be willing to hold that security. Now, how much risk the security adds to the market portfolio depends on its own variance (note: how it covaries with itself) and on how it covaries with other elements of the market portfolio, i.e. other risky assets.

The formalization of this idea yields one of the most famous and useful results in all of finance – the Capital Asset Pricing Model (CAPM) – which can be stated, and demonstrated, as follows:

Theorem 1. (CAPM) *Under the assumptions of classical portfolio theory,*

$$\bar{r}_i = r_f + \beta_i(\bar{r}_m - r_f)$$

where

$$\beta_i = \frac{COV(r_i, r_m)}{VAR(r_m)}$$

is called security i's beta.

Proof. (A bit difficult, but any finance student needs to have seen this at least once.)

Consider the following thought experiment. Assume that we try to get the highest expected return we possibly can by combining some investment α_i in security i, some investment α_m in the market portfolio, and completing the portfolio by investing (possibly short-selling) $(1-\alpha_i-\alpha_m)$ in the risk free asset, **all the while maintaining** the portfolio's risk at σ_m. In other words, we want to choose α_i and α_m to solve:

$$\max \alpha_i \bar{r}_i + \alpha_m \bar{r}_m + (1 - \alpha_i - \alpha_m) r_f$$

subject to:

$$\alpha_i^2 \sigma_i^2 + \alpha_m^2 \sigma_m^2 + 2\alpha_i \alpha_m \sigma_{im} = \sigma_m^2$$

where, for short, $\sigma_i = \sigma_{r_i}$ and $\sigma_{im} = COV(r_i, r_m)$. This is a constrained maximization problem the likes of which you encountered in intermediate microeconomics. To solve this, one writes a Lagrangian:

$$\mathcal{L} = \alpha_i \bar{r}_i + \alpha_m \bar{r}_m + (1 - \alpha_i - \alpha_m) r_f + \lambda \left(\sigma_m^2 - \alpha_i^2 \sigma_i^2 - \alpha_m^2 \sigma_m^2 - 2\alpha_i \alpha_m \sigma_{im} \right),$$

where λ is called a Lagrange multiplier. Then, we differentiate $\mathcal{L}$ with respect to the two choice variables and the Lagrange multiplier λ and set those derivatives to zero, yielding, in the first two cases[1]:

$$\mathcal{L}_{\alpha_i} = \bar{r}_i - r_f - \lambda(2\alpha_i \sigma_i^2 + 2\alpha_m \sigma_{im}) = 0 \quad (1)$$

$$\mathcal{L}_{\alpha_m} = \bar{r}_m - r_f - \lambda(2\alpha_m \sigma_m^2 + 2\alpha_i \sigma_{im}) = 0 \quad (2)$$

Now here's the whole trick to this proof. Up to here, it was all brutal math, now we need to be clever. We know that the market portfolio is efficient. This means that setting $\alpha_m = 1$ and $\alpha_i = 0$ has to solve the problem above. If a solution exists that beats that proposal, it means that there exists a portfolio with the same risk as the market portfolio, but more return. That would contradict the fact that the market portfolio is efficient.

So $\alpha_m = 1$ and $\alpha_i = 0$ must solve both equations above. But note that equation (1) then implies that $2\lambda = \frac{\bar{r}_i - r_f}{\sigma_{im}}$. Plugging that fact into equation (2) and maintaining $\alpha_m = 1$ and $\alpha_i = 0$, we get:

$$\bar{r}_i - r_f = \frac{\sigma_{im}}{\sigma_m^2}(\bar{r}_m - r_f),$$

which is the CAPM equation. □

[1] A technicality here. Those first order conditions must hold when no non-negativity restrictions on the three $\alpha's$ are imposed. So by writing first-order conditions in this fashion, we are allowing in principle for shorting of all assets, no just the risk-free asset. This is without any loss of generality since, as a result of the two portfolio theorem, all risky asset portfolio weights are positive in equilibrium.