Preliminaries

Fixed income

Fixed income basics

- Fixed income, as a field, studies *securities* that stipulate specific financial obligations…
- … in the form of promises to pay interest and to return principal…
- … and the derivatives written on those securities
- Holders of fixed income securities have fully specified cash-flow rights but no control rights
- So these securities are debt claims, the mirror image of equity claims

Largest US issuers

- The US Government
- Government sponsored enterprises (GSEs) and agencies:
  1. Federal National Mortgage Association (FNMA or *Fannie Mae*)
  2. Federal Home Loan Mortgage Corporation (FHLMC or *Freddie Mac*)
  3. Government National Mortgage Association (GNMA or *Ginnie Mae*)
  4. …
- Corporations
- Municipalities
- Commercial and investment banks (via *securitization* facilities)

The running joke

- Every GSE prospectus still stays:

"*Neither the certificates nor interest on the certificates are guaranteed by the United States, and they do not constitute a debt or obligation of the United States or any of its agencies of instrumentalities other than [Fannie or Freddie].*"

- GSEs went into conservatorship the moment they experienced payment difficulties in 2008, and still are

- Securities issued by agencies and GSEs are backed by the full faith of the United States, are priced as such, and we should stop pretending otherwise

Seniority

- *Senior securities* get paid first (within their type) in the event of default, disposition, or reorganization
- *Subordinated/junior securities* get paid after senior securities have been paid
- *Pari Passu*: All creditors at the same level of capital structure are treated as one class

Security

- Secured instruments (mortgages, e.g.) are collateralized by specific tangible assets (real estate, plants, warehouses, machines, airplanes…)
- In theory, secured claims have *absolute priority* over other financial claims, including senior unsecured claims
- To be precise: "*Under the bankruptcy code, secured claims are entitled to receive value equal to the full value of their interest in the collateral before any value is given to holders of unsecured claims, and any priority unsecured claims are entitled to receive the full value of their claims before any general unsecured claims receive any value.*"
- See "A Primer on Second Lien Term Loan Financings", Neil Cummings and Kirk A. Davenport, 2004

Callable/Redeemable securities

- A fixed income security is *callable* if the issuer may *recall/retire* it before maturity
- This is an option which the issuer has the right but not the obligation to exercise
- Terms are specified by the indenture document (aka prospectus):
  1. Lockout period
  2. Call price
  3. Make-whole provisions

Putable securities

- A fixed income security is *putable* if lender(s) may force the issuer to retire it before maturity
- This is an option which the lender has the right but not the obligation to exercise

- Note well: most fixed income contracts feature *acceleration clauses* that are triggered by specific events such as default, but putable securities feature essentially unconditional puts at specific dates

Sinking fund provisions

- A commitment by the issuer to make regular deposits into a trustee-managed fund to be used, eventually, to pay back the issue

Convertible securities

- A fixed income security is *convertible* if the buyer has the right but not the obligation to exchange its debt claims for a fixed number of equity claims
- A traditional bond with a call option

Covenants

- Legally binding agreement between issuer and investors
- *Positive, negative, and financial*
- Typical financial covenants:
  1. Maximum leverage (Debt/EBITDA, say)
  2. Minimum DSCR (EBITDA/Interest)
  3. Minimum rating
- *Incurrence* covenants (as opposed to *maintenance* covenants) are triggered by specific events

Privately traded vs publicly traded securities

- *Private claims:* traded in private markets (restricted and opaque)
- *Public claims:* traded in public markets (much less restricted and much more transparent)

- More narrowly, a security is public if its offer and sale is registered with the *Securities and Exchange Commission* (SEC) and subject to the associated requirements
- Private securities (aka private placements) are securities whose issuer qualifies for and invokes a legal exemption from registration (section 4(a)(2) of the Securities Act of 1933, regulation D…)

What is a public corporation? (take 1)

- A *public corporation* or *publicly traded corporation* is a corporation whose common equity trades in public markets
- Public corporations can and do issue private claims
- Private corporations can and do issue public claims
- Public does **not** mean *listed* on a public exchange

What is a public corporation? (take 2)

"In general, we use the term to refer to a company that has public reporting obligations. Companies are subject to public reporting requirements if they:

1. *Sell securities in a public offering (such as an initial public offering, or IPO);*
2. *Allow their investor base to reach a certain size, which triggers public reporting obligations; OR*
3. *Voluntarily register with us."*

From Investors.gov, i.e. the SEC

Universal principles of debt math

- A debt contract stipulates:
  1. An initial balance/original face value b_0
  2. A frequency of payments and total number T of payments (term to maturity)
  3. An interest rate r_t for each period $t = 1,2, \ldots T$
  4. Payments m_t for each period $t = 1,2, \ldots T$
- Debt algebra:
  1. At a given date, interest due is $b_{t-1}\, r_t$
  2. Balance at the end of date t is $b_t = b_{t-1} + b_{t-1} r_t - m_t$
  3. If $b_T > 0$, balance is due in one *balloon payment*

Interest paid in arrears

- Interest owed at date *t is* $b_{t-1}\ r_t$
- It is owed at the end of period *t*, hence, equivalently on a continuous time scale, at the start of period *t+1*…
- … as interest payment for a loan of size b_{t-1} over the previous period

Simple interest

- We will consider contracts that stipulate an annual interest rate and a payment (hence compounding) frequency
- For instance, a bond with a fixed rate of 10% and a payment frequency of $k=2$ is a bond that pays 5% twice a year
- A bond with payment frequency $k=2$ is called a *semi-annual (S/A) bond*

Practicing debt math: an example

- An untaxed corporation has the option to prepay (*call*) a bond with 5 years to maturity, $\$100M$ in remaining principal, a 10% yearly rate, fixed and monthly payments
- It can replace it with a 5 year bond with the same payment structure but a 9% yearly rate
- It believes rates will fall no further
- Prepayment penalties are 2% of outstanding principal
- Assume not other costs (floatation, overlap…)
- Should it exercise the option?

The algebra

- Current payment is $2,124,704.47
- New payment would be $2,075,835.52, for a monthly saving of $48,869.95
- Appropriate discount rate for the corresponding string of cash flows is 9% (Why?)
- Gross value of refi: $2,354,182.11
- This exceeds prepay costs, *the call option is in the money*, so yes, exercise…
- … as long as you are confident in your belief that rates will fall no further

Deeper option considerations

- Exercising the option kills the option
- If rates fall to, say, 8.5% in two months, the gap between 9% and 8.5% will not suffice to cover prepay costs, so you'll be stuck at 9%
- Had you waited to exercise, you would be able to lower your rate to 8.5%
- What is the value of waiting to exercise an option that is already in the money?
- We will also learn how to answer tough questions like that

Continuous compounding digression

- Simple interest is what people use in practice in most cases so we will do most if not all of our work that way
- In derivations it is sometimes useful to use continuous compounding rather than simple interest
- To be able to follow those arguments, a quick interest algebra digression is order

Simple interest vs. continuous compounding

- If the annualized interest rate is *r* and the number of payments per year (the *frequency)* is 2, then interest paid in a given period is $\frac{r}{2}$ per unit of balance

- Correspondingly, the *effective annual rate* (*EAR*) is:

$$\left(1+\frac{r}{2}\right)^2-1$$

- Generalizing to a frequency of *k* payments per year the effective rate is:

$$\left(1+\frac{r}{k}\right)^k-1$$

- As *k* grows large this converges to $e^r - 1$, which is continuous compounding

- Generalizing, if *r* compounds continuously for *t* years, the effective return is $e^{rt} - 1$

Simple to continuous

A simple interest rate r^s compounded at frequency *k* is equivalent to a continuously compounded interest rate r^c if:

$$e^{r^c} = \left(1 + \frac{r^s}{k}\right)^k$$

i.e.

$$r^c = k \ln\left(1 + \frac{r^s}{k}\right)$$

Continuous to simple

A continuous interest rate r^c is equivalent to a simple interest rate r^s at frequency *k* if:

$$r^s = k\left(e^{\frac{r^c}{k}} - 1\right)$$

Example

- Consider a zero-coupon bond due in one quarter with a face value of *100* and currently selling for *98*. You buy now and hold to maturity.

- What is the simple, annualized, quarterly compounded rate of return on this investment?
- What is the continuously compounded rate of interest on this investment?
- What is the EAR on this investment?

Pro tip: always start with continuous

- In this type of problem, continuous makes life super easy
- We just need $e^{0.25\, r^c} = \frac{100}{98}$ or $r^c = 4\ln(\frac{100}{98}) \approx 8.08$
- Then converting to any sort of simple interest is a breeze
- Quarterly, e.g., is $r^s = 4\left(e^{\frac{8.08}{4}} - 1\right) \approx 8.16$
- Let's look at a real-world version on Bloomberg by calculating the yield on PTON's zero-coupon bond

Some language

- Debt contracts whose balance is zero after T periods ($b_T = 0$) are called *fully amortizing*
- Interest rates can be fixed, vary on a fixed schedule, or according to some other market rate
- Amortization can be negative -- balance can grow -- from one period to the next (if $m_t < b_{t-1}r_t$)

Ex: Bullets (fixed-rate IO bonds)

- For all t:
  1. $r_t = r$
  2. $m_t = b_o r$

- Zero amortization: $b_T = b_0$

Ex: zero-coupon bond

- For all t:
  1. $r_t = 0$
  2. $m_t = 0$
- Zero amortization: $b_T = b_0$

Ex: Floating rate coupon bonds (floaters)

- For all t:
  1. Initial rate: r_0
  2. At reset, $r_t^* = index_t + QM$
  3. QM is the typically fixed quoted margin
  4. r_t adjust towards r_t^* subject to caps and floors, both absolute and on adjustment steps
  5. $m_t = b_o r_t$

- Zero amortization: $b_T = b_0$

- Libor was the most typical index but we have now moved towards alternative reference rates like CMT or SOFR

Ex: Inverse floaters

- For all t:
  1. $r_t = \bar{r} - index_t$ where $\bar{r}$ is some fixed reference
  2. caps and floors

- Those, together with duration/maturity mismatch, killed Orange County finances in 1994

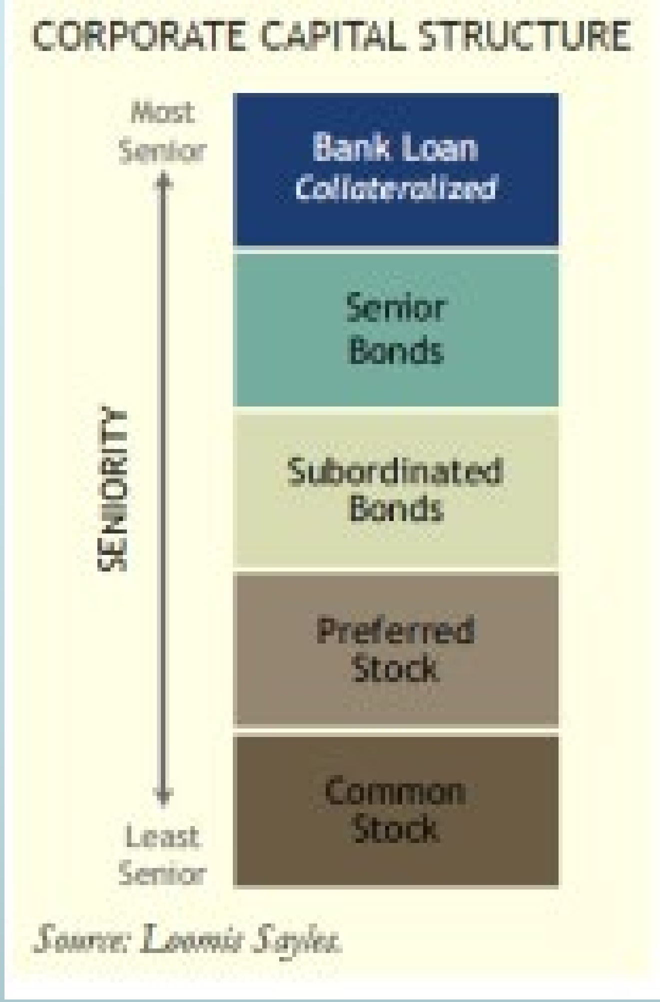
Ex: Fixed rate, fixed payment debt

- For all t:
  1. $r_t = r$
  2. $m_t = m$
- Fully amortizing: $b_T = 0$
- What must m be? (Fixed annuity formulae)
- $m = b_0 r / (1 - (1 + r)^{-T})$

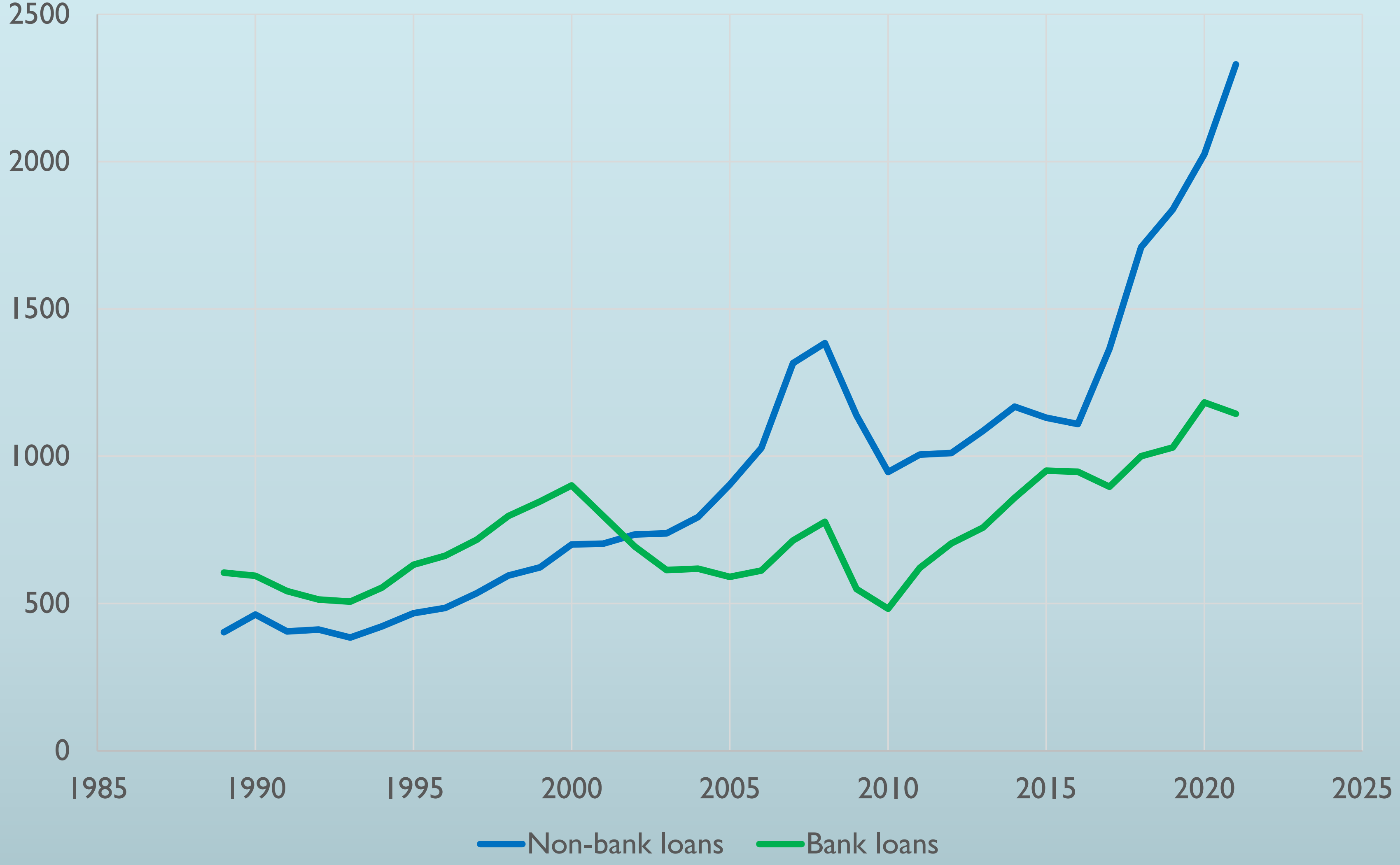
Term loan facilities (*Project Financing*)

Term loan A	Term loan B
Bank ("pro rata") loans issued to typically IG corporate borrowers	Syndicated loans often made to below IG-borrowers, institutional lenders active
Bespoke	Bespoke to some extent
Senior, first lien	Senior, first lien
Short tenor (5-7 years typical)	May have longer tenor
Floating rates	Floating rates
Allow for prepayment with little to no penalties	Allow for prepayment with little to no penalties
Significant amortization	Little to no amortization
Tight covenants, especially w.r.t. operations and leverage	Covenant-lite

The traditional capital stack

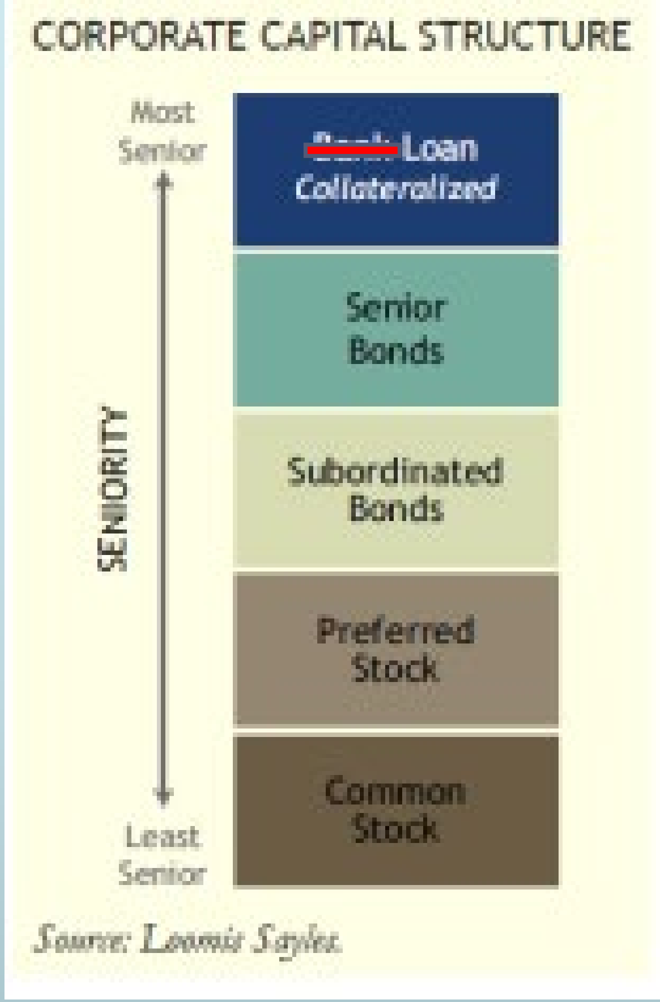


Who needs banks any more?



Notes: $bn, total outstanding, flow of funds data, bank loans are loans by depository institutions to NCBs

Capital stack, 2023 version



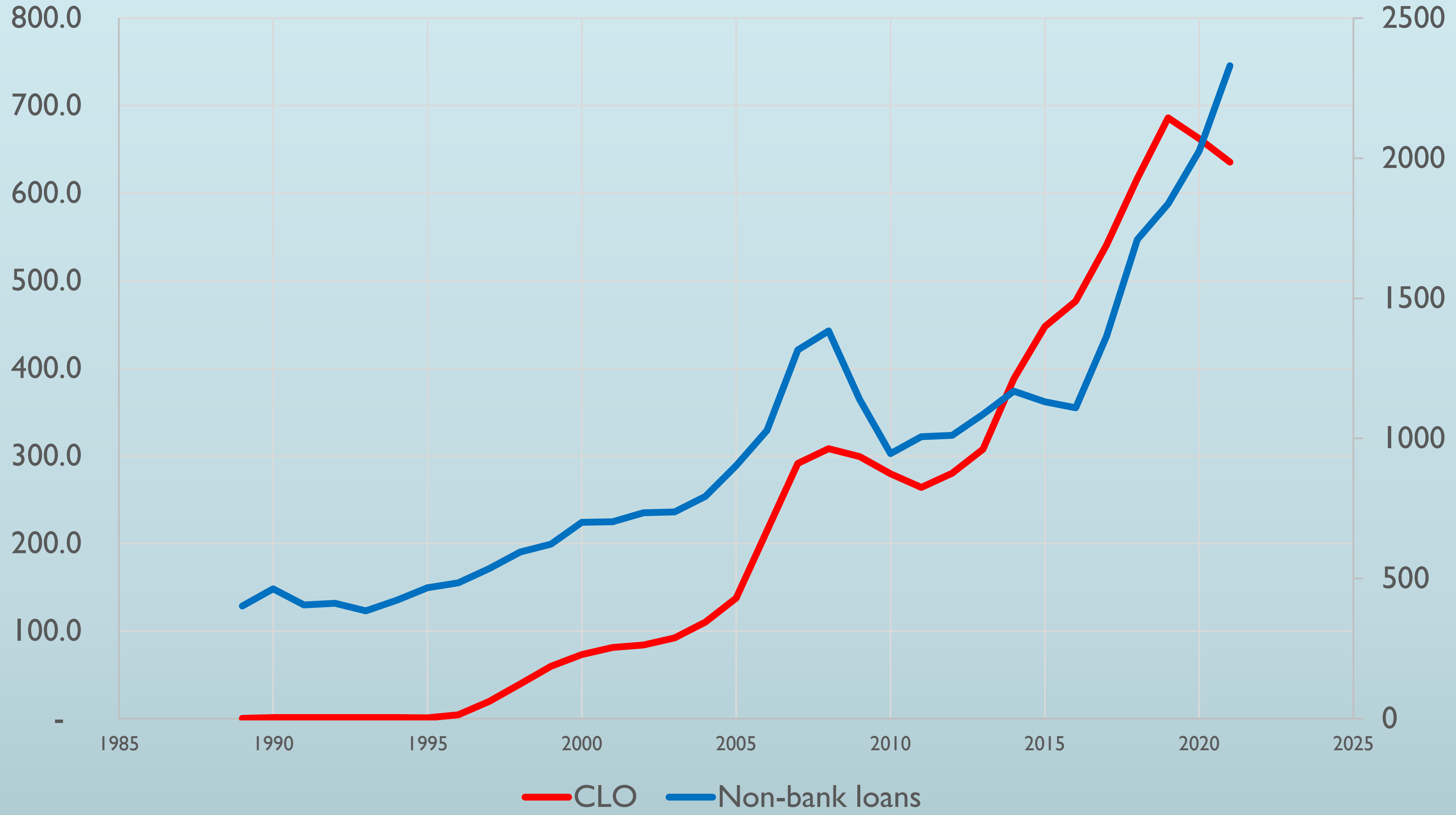
Revolving credit facilities (*revolvers*)

- Credit lines in exchange for fees
- "Deal sweeteners," usually combined with TLs

Collateralized Loan Obligations (CLOs)

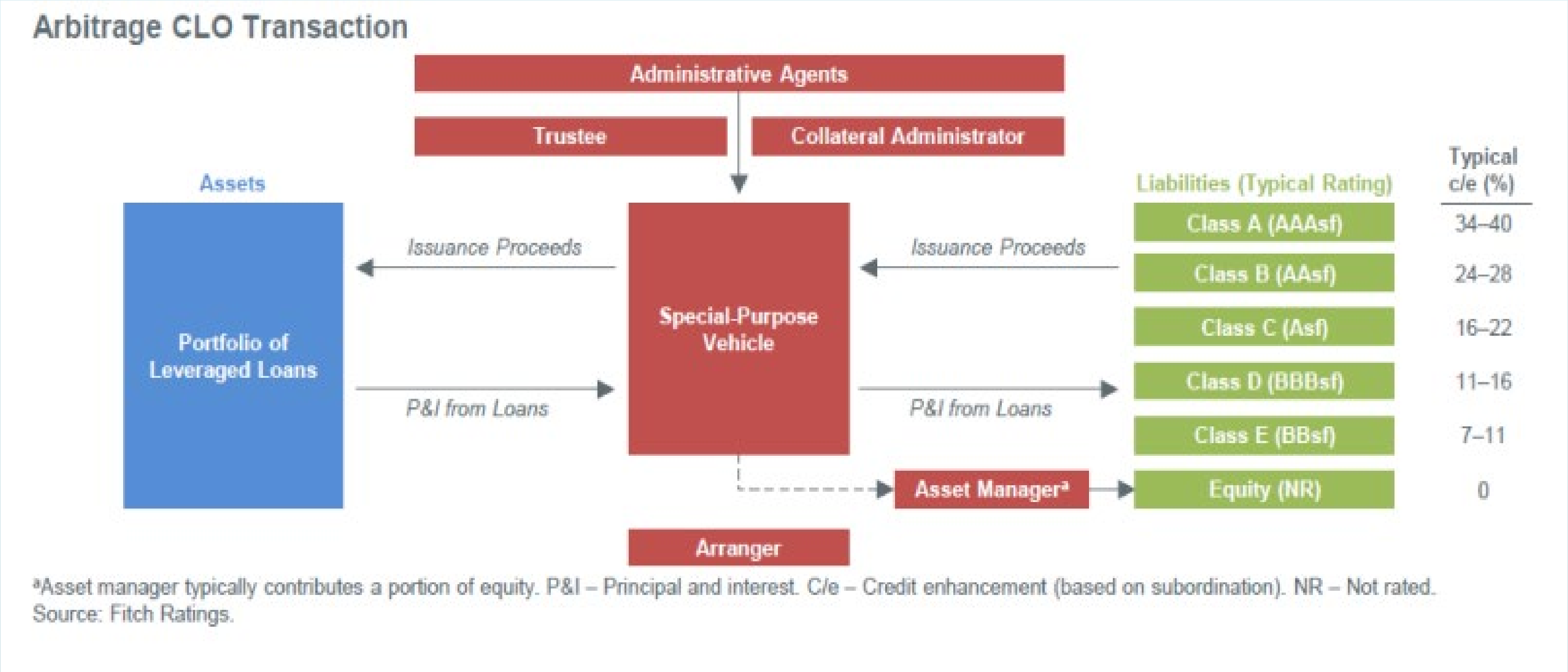
- Increasingly, term loans B (C, D,…) are pooled into special purpose vehicles (*SPVs/SIVs*)…
- And the resulting cash-flows are tranched into securities that cater to investors with different needs
- We've seen this movie before, and we'll discuss all that in chapter 7

Securitization has mushroomed



- Notes: $bn, total outstanding, SIFMA, flow of funds. Loans to NCBs, excluding mortgages

Looks familiar?



My favorite part is the clever use of the word "arbitrage" here, marketing is one hell of a drug:

"The transaction is referred to as "arbitrage" because it aims to capture the excess spread between the portfolio of leveraged bank loans (assets) and classes of CLO debt (liabilities), with the equity investors receiving any excess cash flows after the debt investors are paid in full."

Source: NAIC's CLO primer

Are term loans securities?

- That case has been made legally on several occasions by plaintiffs seeking the stronger protection of federal and state ("*blue sky*") security laws (see here, e.g.)

- Securities, broadly and intuitively, are tradable financial contracts

- Legally, things are much, much more complicated, like always

- Loans, even syndicated ones, so far, have not been deemed to be securities

- And yet we will study them in this "Analysis of FI securities" class because:
  1. They walk like a duck, case law notwithstanding
  2. You can't call yourself a fixed income person if you don't know term loans inside and out
  3. They do become used to create FI securities once part of a CLO pool

What is a security?

- Section 2(1) of the 1933 Act defines a security as follows:

"The term "security" means any note, stock, treasury stock, bond, debenture, evidence of indebtedness, certificate of interest or participation in any profit-sharing agreement, collateral-trust certificate, preorganization certificate or subscription, transferable share, ***investment contract****, voting-trust certificate, certificate of deposit for a security, fractional undivided interest in oil, gas, or other mineral rights, or, in general, any interest or instrument commonly known as a "security", or any certificate of interest or participation in, temporary or interim certificate for, receipt for, guarantee of, or warrant or right to subscribe to or purchase, any of the foregoing."*

The fundamental equation of debt design

- Full amortization means:

$$b_T = 0, \quad \text{or, equivalently,} \qquad b_0 = \sum_{t=1}^{T} \frac{m_t}{(1+r)^t}$$

- More generally:

$$b_0 = \sum_{t=1}^{T} \frac{m_t}{(1+r)^t} + \frac{b_T}{(1+r)^T}$$

- Absent side payments, r is the loan's IRR <u>if all payments are made</u>, i.e. the YTM on the debt contract

Market value vs book value

- The book value of a bond is its face value b (we'll often write *FV* too)

- Market value is what the bond would sell for in the market:

$$MV = \sum_{t=1}^{T} \frac{m_t}{(1+y)^t} + \frac{b_T}{(1+y)^T}$$

where y is the *yield-to-maturity* investors are currently requiring from the bond

Market value drivers

- Tautologically:

$$y = r^F + RP$$

- Ergo, two things move market values:
  1. The general level of interest rates (*market conditions*)
  2. Spread changes caused by category-wide or issuer specific reasons

- Trivially, $MV < FV$ if and only if $y > r$, and vice versa

Illustration: floaters

- Floaters adjust to movements in r^F so, at least around resets, floaters subject to no credit risk should trade close to par
- But floaters do not adjust to movements in *RP*
- If spreads over benchmarks narrow or broaden, the market value of outstanding floaters will deviate from par

Discount margins (DM)

- A standard measure of par deviations for floaters, computed at resets, is the discount margin

- Letting MV be the invoice price of a semi-annual floater with N payments left at reset:

$$MV = \sum_{n=1}^{N} \frac{FV \times \frac{index_1 + QM}{2}}{\left(1 + \frac{index_1 + DM}{2}\right)^n} + \frac{FV}{\left(1 + \frac{index_1 + DM}{2}\right)^N}$$

- Note that $MV < FV$ if and only if $DM > QM$
- Note also the assumption in this calculation that the index will stay at its current value ($index_1$) to maturity

Fixed payment example

- $100K$, monthly payments, 10 years, $r = 7\%/12$

1. With full amortization: $m = \$1{,}161.08$
2. With 30K balloon: $m = \$\,987.76$

Graduated payment example

- A corporation wants to finance 50% of a $\$1M$ investment with some debt
- Cash-flows are expected to increase by 0.5% a month over ten years, so the corporation would like debt payments to match that profile
- A lender is willing to do it with a 7% annual interest rate and a fully amortizing contract
- What is the payment schedule?
- Answer: fundamental equation of debt design

YTM, YTC, YTW

- *Yield-to-maturity:* Investor's IRR assuming all payments are made to maturity

- *Yield-to-call:* Investor's IRR assuming a bond is redeemed at the first call date (after figuring call price and/or make-whole provisions)

- *Yield-to-worst=min(YTC,YTM)*

Example

- Consider a four-year callable bullet with exactly 4 years to maturity, a coupon rate of 8% and 8 S/A payment left
- The bond is callable in 2 years (just after payment 4 is made) at a call price of 102 per 100 of principal.
- The redemption value is 1, 000 and the current market price is 1, 050

- What are the bond's YTM, YTC, and YTW?

Spot-yield curve (the z's)

- A curve that shows the annualized ytm's of zero-coupon risk-free bonds
- The fundamental building block of all fixed-income thinking…
- … because all fixed income securities, in the end, are packages of zeros, and must be priced accordingly

From z's to discount factors

- Spot yields give us discount factors:

$$DF(t) = \left(1 + \frac{z(t)}{k}\right)^{-kt}$$

- That is the market price of a *1$* risk-free payoff in *t* periods assuming a compounding frequency of *k*

- But then we can price any risk-free security that pays *CF(t)* at a sequence dates *t* simply as:

$$\sum_t CF(t) \times DF(t)$$

STRIPS

- STRIPs let investors hold and trade the individual interests and principal components of eligible treasuries
- Created by intermediaries but registered at the Treasury and still backed by the full faith of the US government
- Zero-coupon treasuries "stripped" from treasury bullets
- In principle, a direct way to look up spot rates BUT:
  1. Liquidity not great on STRIP market
  2. Sparse maturities, much interpolation needed
  3. Tax treatment issues
- So we usually turn elsewhere for our zero's

Bootstrapping the treasury curve

- Another way to get our z's is to get them from the price of Treasury bonds and notes using a process called bootstrapping
- We will learn it in chapter 3 but it too leaves us with sparse maturities and so would require heavy interpolation
- So once again we turn elsewhere, namely Eurodollar futures and the LIBOR/SOFR swap market

Par yield

- Given zero-coupon yields, the coupon rate that makes market value of a risk-free bond equal to par
- Example: Zero-coupon spot rates are 1, 2, and 3 percent at respectively, 6m, 1y, 1.5y. What is the 1.5y par yield?
- Letting c denote the par yield, it solves:

$$\frac{100\frac{c}{2}}{1+0.5\%}+\frac{100\frac{c}{2}}{(1+1\%)^2}+\frac{100+100\frac{c}{2}}{(1+1.5\%)^3}=100$$

- So:

$$c=2\times\frac{1-D(1.5)}{D(0.5)+D(1)+D(1.5)}$$

- Now, the key: par yields are de facto swap rates

Plain-vanilla interest rate swaps

Fixed leg buyer **Floating leg buyer**

$Notional(A) \times floating\ rate\ (\tilde{r}_t)$ →

← $Notional(A) \times Swap\ rate\ (r)$

Swap rates are par yields

- At origination r is set so that PV(fixed leg)=PV(floating leg)
- But, at least roughly, PV(floating leg) = A – PV(A at maturity)…
- … while PV(fixed leg) = PV(Ar to maturity)
- Ergo PV(Ar to maturity) + PV(A at maturity)= A
- Swap rates are par yields!
- Better yet, they are basically risk-free par yields given the parties involved!
- Because swap markets are more active and complete than treasury markets (= require less interpolation), swap-implied spot yields are used in lieu of treasury-implied spot yields in most fixed income calculations
- Plus US treasury yields are "special" (look up *convenience yield)*

Why is PV(floating leg) = A – PV(A at maturity)?

- The floating leg is *investable:*
  1. *Invest A at floating rate to maturity*
  2. *Recover A in the end*

- That simple strategy costs A today and pays $\{A\tilde{r}_t\}_{t=1}^{T}$ plus *A* at maturity

- So *A=PV(floating leg) +PV(A at maturity)*

- Not just true at origination, true at every reset

- Argument only limited by:
  1. Investability assumption (transaction costs)
  2. Counterparty risk

- So it doesn't apply well to corporate floaters or Real Estate swaps, among others, but applies very well, if not perfectly, for LIBOR/SOFR swaps…

But then we can use swap rates to get spot rates!

- Assume S/A swap rates hence par yields are 1%, 2%, and 3% over the next 18 months, what are spot yields?

- First one is easy: 1%

- Second one, we know:

$$100 = \frac{1}{1 + \frac{1\%}{2}} + \frac{101}{\left(1 + \frac{z(1)\%}{2}\right)^2}$$

- And so on for deeper maturities
- Bootstrapping!

Practical difficulties

- Swaps compound quarterly where T-bonds compound semi-annually
- With different day-count conventions too in some conversion cases
- That's why we use Bloomberg to handle those details
- <SWPM>

Example

- For short swaps with one payment, Bloomberg does this:

$$DF(s,t_i) = \left(1 + R_{t_i}^{mkt} \times \frac{\tau(s,t_i)}{360}\right)^{-1}$$

$$R(s,t_i) = \frac{365}{\tau(s,t_i)} \times \log\left(\frac{1}{DF(s,t_i)}\right)$$

$DF(s,t_i) =$ discount factor from t_i to s

$R(s,t_i) =$ zero or spot rate from t_i to s

$R_{t_i}^{mkt} =$ market swap rate at t_i

$\tau(s,t_i) =$ day count

- Then, for longer maturity swaps, bootstrap along the lines we studied above

One more detail

- Usually the swap curve is supplemented by eurodollar data, especially at short tenors
- Eurodollar futures: cash-settled, marked-to-market LIBOR futures contract where, by design, contract value falls (rises) by $25 when the LIBOR rate rises (drops) by a basis point
- A very liquid and active market that can be used:
  1. To lock in LIBOR rates
  2. To bet on future LIBOR rates
- Eurodollar rates give the expected value of LIBOR at various horizons
- So we use them to bootstrap discount rates

Day-count convention

- Between coupon dates interest accrues as time passes

- Day-count conventions are used to measure the fraction of time elapsed between coupon dates

- Two main conventions in the US:
  1. 30/360 US
  2. Actual/Actual ICMA (Treasuries)

- All captured by yearfrac() in Excel*, Python, Matlab…

* Excel, inexplicably, does actual/365 instead of actual/actual ICMA, but I expect that to be fixed at some point. Matlab, not to mention Python and the like, do things properly of course.

30/360 US

- Take two dates (M_1, D_1, Y_1) and (M_2, D_2, Y_2), how many "years" are they apart?

- Day count factor (DCF) =

$$\frac{360 \times (Y_2 - Y_1) + 30 \times (M_2 - M_1) + D_2 - D_1}{360}$$

- With a few annoying adjustments we need to make:
  1. If $D_2 = 31$ and $D_1 = 30\ or\ 31$, then $D_2 = 30$
  2. If $D_1 = 31$, then $D_1 = 30$
  3. February adjustment

Actual/Actual ICMA

- Day count factor =

$$\frac{Days(Date1, Date2)}{Days\ in\ coupon\ period \times Frequency}$$

Accrued interest

$$Accrued\ interest = Day\ count\ factor^{**} \times Principal \times Coupon\ rate^{*}$$

where, note:

** DCF is measured since last interest payment

* Use annualized coupon rate

Clean and dirty prices

- *Dirty/invoice price* is the price actually paid for the bond on the *settlement date*

- But what is quoted on Bloomberg (and all similar platforms) is the *clean price* which equals:

dirty price – accrued interest

Clean and dirty prices

$$Clean\ price =$$

$$\left[\frac{redemption}{\left(1+\frac{YTM}{k}\right)^{N-k*DCF}}\right]$$

$$+\left[\sum_{n=1}^{N}\frac{redemption \times \frac{rate}{k}}{\left(1+\frac{YTM}{k}\right)^{n-k*DCF}}\right]$$

$$-redemption \times rate \times DCF$$

- *Notes: Here, k is the frequency, N is the total number of payments, and n indexes periods. DCF is the day count factor since the most recent coupon.*

Put another way…

$$Clean\ price + accrued\ interest =$$

$$Dirty\ price =$$

$$\left[\frac{redemption}{\left(1+\frac{YTM}{k}\right)^{N-k*DCF}}\right]$$

$$+\left[\sum_{n=1}^{N}\frac{redemption \times \frac{rate}{k}}{\left(1+\frac{YTM}{k}\right)^{n-k*DCF}}\right]$$

Tick size and quotation

- Treasury notes are quoted relative to par, in 32nd of a dollar (*tick size*):

ISSUE	BID	ASK	CHANGE	YIELD
6 1/2 8/15/05-N	105.08	12	+3	5.57

- This note (N) was issued with a coupon rate of 6.5% and a maturity of 8/15/05
- Bid price is $(105+8/32), ask price is $(105+12/32)
- Yield is YTM at ask price
- Third "decimal" added as needed: + for ½, 2 for ¼, 6 for ¾ of a 32nd

Tick size and quotation

- Treasury bills are quoted as discount to par on an annual Actual/360 day basis):

ISSUE	BID	ASK	CHANGE	YIELD
12/3/98	5.08	5.06	-.03	5.26

- 12/3/98 is the maturity date
- Assume we are 169 days to maturity in example above, the bid price of the bill, per $10k of par is:

$$\$10{,}000 - \frac{(508 \times 169)}{360} = \$9{,}761.52$$

Main risks of investing in fixed income

1. Interest rate risk
2. Re-investment risk
3. Inflation risk
4. Default/credit risk
5. Call/prepayment risk
6. Liquidity risk
7. Exchange rate risk
8. “Volatility risk”

Duration and convexity

- 2nd order Taylor expansion for any smooth function:

$$df = f(x + dx) - f(x) \approx f'(x)dx + \frac{1}{2}f''(x)(dx)^2$$

- Let $V(y)$ be a bond value given bond yield y,

$$\frac{dV}{V} \approx -D\ dy + \frac{1}{2}C(dy)^2$$

where D is *modified duration* and C is *convexity*

Macaulay duration and modified duration

- Given k coupons per year and a total of N payments:

$$V = \sum_{n=1}^{N} \frac{CF_n}{\left(1 + \frac{y}{k}\right)^n} = \sum_{n=1}^{N} PV_n$$

- Macaulay duration:

$$McD = \sum_{n=1}^{N} \frac{PV_n \times \frac{n}{k}}{V}$$

- Modified duration:

$$D = \frac{McD}{1 + \frac{y}{k}}$$

Duration facts

- Modified duration increases with maturity, falls as the yield rises, and also falls as the coupon rate rises
- Macaulay duration of a zero of maturity *T is T*
- Modified duration of that zero is $\frac{T}{1+\frac{y}{2}}$
- Duration of portfolio of fixed income instruments is the weighted average of the duration of each element of the portfolio, with weights proportional to market values
- Risk-free floaters (should roughly) return to par at each reset so they are like very-short term zeros
- Their duration is the time to the next reset, i.e., super short
- Swapping fixed (long-duration) for floating (short-duration) is the easiest way to manage one's duration

Dollar duration measure

- "Bloomberg risk":

$$\frac{D}{100} \times V$$

- Dollar duration (*DV01)* is the risk per basis point change in rates:

$$\frac{D \times V}{10{,}000}$$

- *Note: duration and convexity are always "(weigthed) average", while dollar durations and convexity are additive. If you buy the same bond twice, duration and convexity are unaffected, but dollar duration doubles. More on this in two slides.*

Convexity (semi-annual bonds, N payments)

$$C = \frac{1}{\left(1+\frac{y}{2}\right)^2} \sum_{n=1}^{N} \frac{CF_n \times \left(\left(\frac{n}{2}\right)^2 + \frac{n}{4}\right)}{V\left(1+\frac{y}{2}\right)^n}$$

$$= \frac{1}{\left(1+\frac{y}{2}\right)^2} \sum_{t=\frac{1}{2},1,\frac{3}{2},\dots\frac{N}{2}} \frac{PV_t \times \left(t^2 + \frac{t}{2}\right)}{V}$$

Convexity facts

- Convexity increases (in a convex fashion) with maturity, falls as the yield rises, and generally falls as the coupon rate rises
- Convexity of a zero of maturity T *is* $\frac{T^2+\frac{T}{2}}{\left(1+\frac{y}{2}\right)^2}$ which is $O(T^2)$
- Convexity of portfolio of fixed income instruments is the weighted average of the duration of each element of the portfolio, with weights proportional to market values
- Some bonds (callable bonds, e.g.) can feature negative convexity
- Convexity is good so more convex bonds, all else equal, should in general be more expensive
- This is why we generally expect the term structure to be concave

Portfolio duration and convexity

- Consider two bonds with modified duration D_1, D_2 respectively, convexity C_1, C_2, and market value MV_1, MV_2
- MV weights are $\omega_1 = \frac{MV_1}{MV_1 + MV_2}$ and $\omega_2 = \frac{MV_2}{MV_1 + MV_2}$
- Portfolio duration is $D = \omega_1 D_1 + \omega_2 D_2$
- Portfolio convexity is $\mathrm{C} = \omega_1 C_1 + \omega_2 C_2$
- On the other hand, dollar durations simply add

Duration times spread

- Hard to compare bonds that feature very different spreads (over risk-free benchmarks)

- So common practice is to measure (first-order) rate sensitivity as:

$$DTS = Duration \times Spread$$

- Premise is that spread ratios ought to be stable, that relative spread changes are more reasonable to expect than absolute rate changes

Spread duration

- Percentage change in the price of bond caused by a 100 basis point (1%) change in spread
- For fixed-rate instruments, same as modified duration
- For floaters, very different

Ex: Floating rate coupon bonds (floaters)

- For all t:
  1. Initial rate: r_0
  2. At reset, $r_t^* = index_t + QM$
  3. QM is the typically fixed quoted margin
  4. r_t adjust towards r_t^* subject to caps and floors, both absolute and on adjustment steps
  5. $m_t = b_o r_t$

- Zero amortization: $b_T = b_0$

- Libor was the most typical index until last year but we have now moved towards alternative reference rates like CMT or SOFR

Effective duration

- Same as modified duration, except that you take into account the fact that expected cash-flows also change when rates change

Key rate duration

- All the duration notions we have discussed so far are about the sensitivity of a bond (or a portfolio) to its own yield
- This does not take into account the term structure of the relevant yield curve
- A key rate duration measures the effect of a change in yield required a specific maturity
- And enables investors to see where their portfolio are most exposed
- Note: Modified duration = sum of key rate durations

Credit-default swaps, a preview

Protection/CDS buyer **Protection/CDS seller**

$$Premium = Notional(A) \times Swap\ rate\ (\kappa)$$

If "credit event"

$$Payment = Face\ Value\ of\ oustanding\ debt - Market\ Value$$
$$= Loss\ given\ default\ (LGD)$$

CDS rates are risk-premia

- Intuitively (we will prove it under premise of perfect markets), CDS rates should match the spread between the yield on the underlying and the risk-free counterpart

- Indeed, both are compensation for probability of default times loss given default

- Two big, practical applications of this fundamental fact:
  1. Synthetic CDOs *(Who needs to issue securities any more? Just fake it with purely nominal CDS contracts)*
  2. CDS basis convergence trades

Convertible bonds

- A *convertible bond* gives the holder the right to convert her bond investment into equity at an agreed-upon *conversion price* and/or *conversion ratio*

- Example: Consider a 6-month convertible bond with face-value $1,000, S/A coupon rate of 10%, and a conversion price of $25. No issuer call option, no default.

- The conversion ratio is $\frac{1000}{25} = 40$ shares per bond

Convertible bond valuation

- Assume the share price 6 months from now is either \$20 or \$30, each with equal ***risk-neutral probability*** and that the 6-month risk-free (z) rate is 5% (annualized as always)
- What is the value of the bond?
- If the holder does not convert in 6 month, they get \$1,050
- If they do convert they get $40 \times$ share price plus accrued interest
- Obviously the option is exercised when and only when the share price is \$30

Fundamental theorem in Finance

No arbitrage

$$\Longleftrightarrow$$

$$q = \frac{E^*(X)}{1 + r^F}$$

where the expectation* is with respect to a synthetic probability distribution called the risk-neutral probability and r^F is the risk free rate

Most of modern finance prices assets by estimating the RNP first and then pricing assets as if agents were risk neutral

This is especially true in fixed income

Convertible bond valuation (2)

- The value of the bond is (should be, to be precise)

$$\frac{\frac{1}{2}\times 1050+\frac{1}{2}\times(40\times 30+50)}{1+\frac{5\%}{2}} = 1{,}121.95$$

- Another way to write this is:

$$\frac{1{,}050}{1+\frac{5\%}{2}} + \frac{\frac{1}{2}\max(40\times 20-1000{,}0)+\frac{1}{2}\max(40\times 30-1000{,}0)}{1+\frac{5\%}{2}}$$

Convertible bond valuation (2)

- The value of the bond is (should be, to be precise)

accrued interest

$$\frac{\frac{1}{2}\times 1050+\frac{1}{2}\times(40\times 30+50)}{1+\frac{5\%}{2}} = 1{,}121.95$$

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$$\frac{1{,}050}{1+\frac{5\%}{2}} + \frac{\frac{1}{2}\max(40\times 20-1000{,}0)+\frac{1}{2}\max(40\times 30-1000{,}0)}{1+\frac{5\%}{2}}$$

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Value of the bond without the conversion feature
(*investment value*)

Convertible bond valuation (2)

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$$\frac{\frac{1}{2}\times1050+\frac{1}{2}\times(40\times30+50)}{1+\frac{5\%}{2}} = 1{,}121.95$$

- Another way to write this is:

Value of the call option to convert, a.k.a warrant

$$\frac{1{,}050}{1+\frac{5\%}{2}} + \frac{\frac{1}{2}\max(40\times20-1000,0)+\frac{1}{2}\max(40\times30-1000,0)}{1+\frac{5\%}{2}}$$

Value of the bond without the conversion feature (*investment value*)

Practical aside from corporate finance

- Decomposing a convertible into straight debt and the value of the warrant is a standard exercise in corp fin, e.g. in the context of measuring a company's WACC

- Process is simple:
    1. Discount coupons in principal at the YTM the market requires on debt of this type absent options
    2. Get the value of the call as a residual
    3. Get the beta of the call as $\beta \times \frac{S}{C} \times \delta$ where β is the stock's beta, *S* is the current stock price, *C* is the value of the call, and δ is the call's delta, estimated using Black-Scholes *(see proof next slide)*
    4. Put it all together to get the overall impact of convertible on cost of capital

- This is all well and good for corp fin but this presumes that the warrants are properly priced by the market

- Fixed income investors look for and look to exploit warrant mispricing

Approximate warrant beta formula

- Let S and C be the current stock and call prices, δ be the call's delta, while β_E and β_C are the stock and the call's betas, respectively
- Then over a given holding period:

$$\beta_C = \frac{COV(r_C, r_M)}{VAR(r_M)} = \frac{COV\left(\frac{dC}{C}, r_M\right)}{VAR(r_M)}$$
$$= \frac{COV\left(\frac{\delta dS}{C}, r_M\right)}{VAR(r_M)}$$
$$= \frac{\delta S}{C} \times \frac{COV\left(\frac{dS}{S}, r_M\right)}{VAR(r_M)}$$
$$= \beta_E \times \delta \times \frac{S}{C}$$

- Notes:
  1. Only first order stock price effect is captured
  2. This presumes a good estimate of δ, usually based on Black-Scholes

Pure convertible bond arbitrage

- Assume that immediately prior to the final coupon payment, the share having already moved up to $30, the bond trades at $1,150
- Then buy the convertible, convert and sell the resulting shares gives you a pure arbitrage
- But that is not what hedge funds mean by convertible arbitrage, a practice made popular by Ed Thorp, the spiritual father of all quants
- *Convertible arbitrage* in the way quants use the expression is not an arbitrage, it is a *convergence trade*

Ed Thorp's version

- If value of the convertible bond is less than value of the non-convertible equivalent plus the value of the call option (the value of the *warrants*):
  1. *Buy the convertible bond*
  2. *Short the stock*
- The number of shares (the *hedge ratio*) is set to immunize the impact of movements in the share price on the bond, this is called *delta hedging*
- Hedge ratio is continuously/dynamically adjusted by buying or selling more stock

No arbitrage

- Tons of risks remain:
  1. *Interest rate risk (can be hedged in standard ways)*
  2. *Default risk (negative gamma/convexity in the large)*
  3. *Risk that mispricing will worsen rather than increase, like it does during financial crises*

Positive gamma?

- When price goes up, bond's delta goes up **<u>locally</u>**, and vice versa
- The rate of change in delta as prices change is called gamma
- When the price goes up the bond's value goes up both because the probability of conversion goes up and the conversion price goes up
- In principle, this makes Ed Thorpe's position **<u>locally</u>** convex

Bond value vs stock price

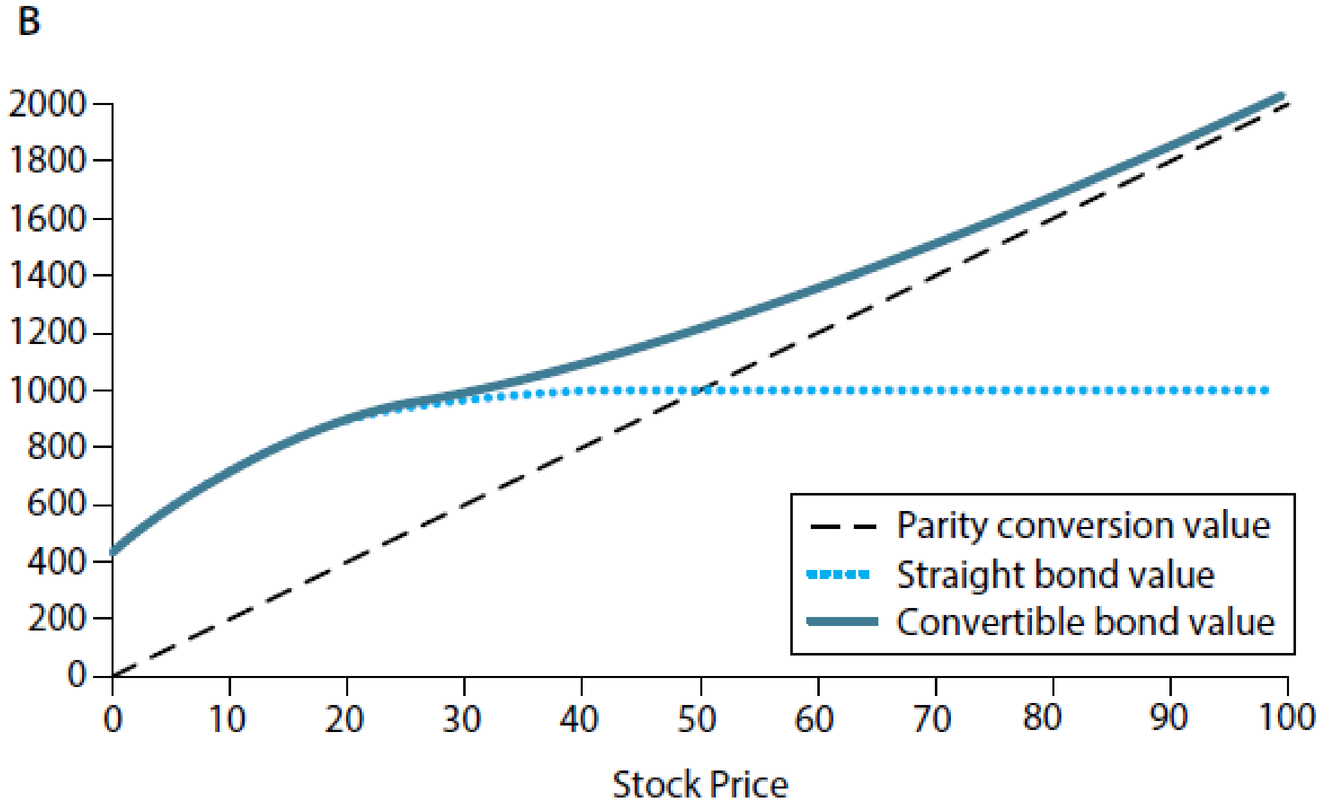


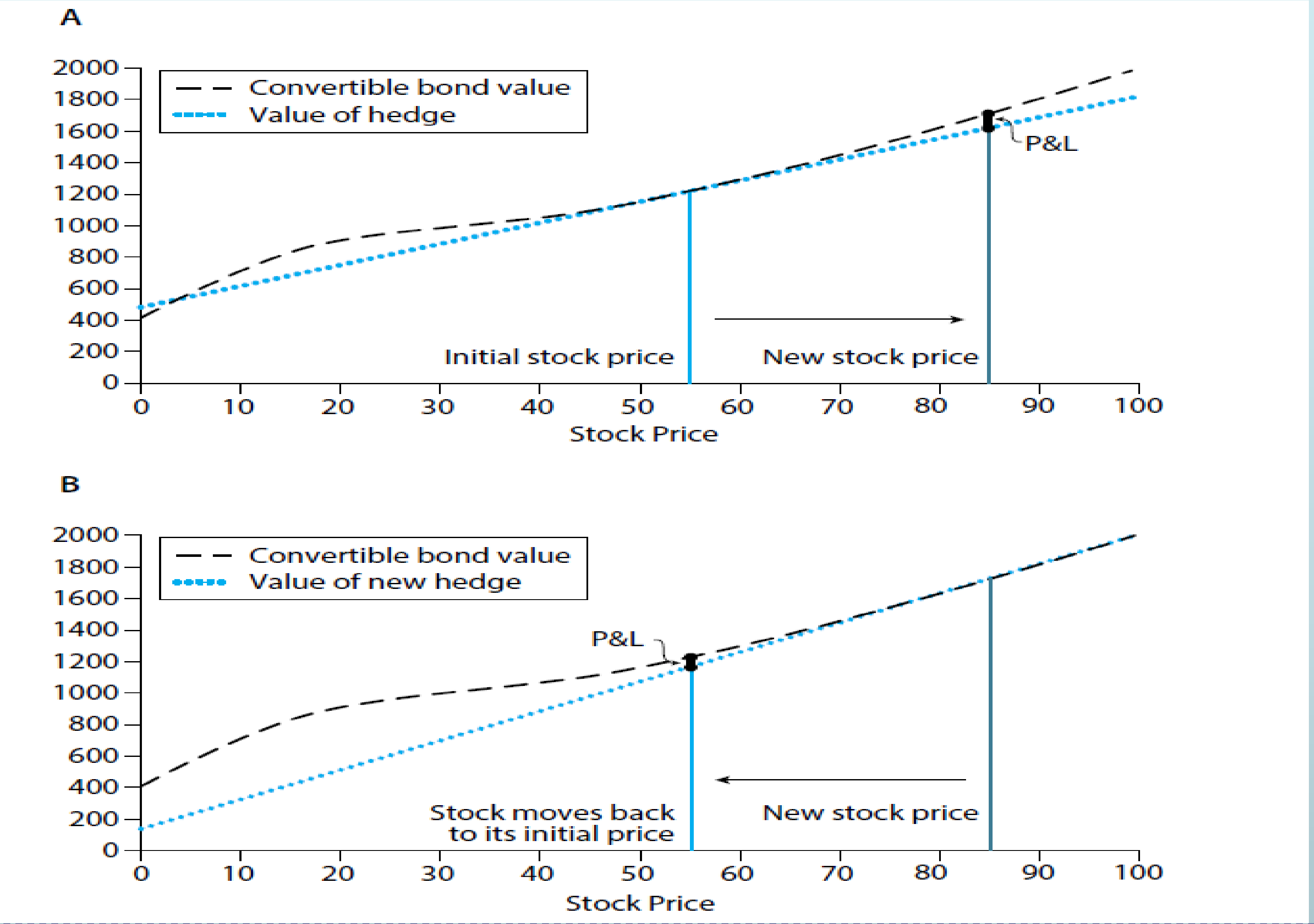
Figure 15.3. How the value of a convertible bond depends on the firm value and stock price.

Panel A. Convertible bond value vs. firm value.

Panel B. Convertible bond value vs. stock price.

- Source: Pedersen's "Inefficiently efficient," a great book

Dynamic hedging



- Source: Pedersen's "Inefficiently efficient," a great book

Stat arb

- Mine for and exploit statistical relationships, usually centers around some reversion to the mean argument
- Example: we expect yields on similar instruments for KO and PEP to be *co-integrated*
- If an exogenous event (large trade, say) nudges the relationship, enter a *pair trade* until shock is dissipated

Buy KO CB USD SR
Sell PEP CB USD SF
96) Actions
97) Edit
Spread Analysis
BMRK Mid YTM
BMRK Mid YTM 03/18/2021 - 03/18/2022
Regression Corr 120
Mult 1.0 Const 0.0
Mult 1.0 Const 0.0 Normalize by Factor 100.0 Calc % Local CCY
1D 3D 1M 6M YTD 1Y 5Y Max Daily Table
Add Studies or Events
Edit Chart
Mid Yield To Maturity
BH178254 Corp 2.648
AP448686 Corp 2.664
Hi: 2.752
Hi: 2.806
67.66%
74.90%
2.648
Low: 1.146
Low: 1.051
Spread Summary
Last -1.6643
Mean -2.7637
Off Avg 1.0995
Median -2.7729
StDev 5.204
StDev from Mean 0.2113
Percentile 57.9365
High 03/10/22 11.4274
Low 04/12/21 -14.3659
BH178254 Corp - AP448686 Corp -1.6643
Hi: 11.4274
Low: -14.3659
-1.6643
Mar Apr May Jun Jul Aug Sep Oct Nov Dec Jan Feb Mar
2021
2022

Speed of convergence

- Consider the following model of the spread:

$$S_{t+1} = \gamma S^* + (1 - \gamma)S_t + \epsilon_{t+1}$$

where ϵ is noise and γ is the speed of adjustment

- If the spread has a tendency to return to some mean then γ should estimate to a number between 0 and 1
- The closer to 1 the faster the speed of adjustment
- A common stationarity test (Dickey-Fuller) involves testing the hypothesis that $\gamma = 0$

Treasury basis trade

- Assume a Treasury zero is available for P in the spot market at F in the futures market
- Arbitrage requires $P(1+r) = F$ with r the borrowing cost over the investment horizon
- But, typically, $P(1+r) < F$ and the gap is called the treasury basis
- Hedge funds arbitrage the difference
- Risks:
  1. Roll-over
  2. Margin calls
  3. Fire sales
- Example: Trump's tariff saga

Fixed income strategies

- Macro:
  1. Economic activity
  2. The level of interest rates
  3. The shape of interest rates
- Industry/sector *allocation*
- Fundamental credit analysis (*selection*)
- Arbitrage/convergence trades: market neutral in principle, but watch out for the steamroller