Fundamentals of asset pricing

Real estate finance

Asset pricing models

- Stylized worlds in which fundamental asset values can be calculated exactly
- We are going to make a number of heroic assumptions
- These stylized models enable us to:
  1. emphasize and understand fundamental determinants of asset value
  2. derive asset pricing rules that serve as useful benchmarks in practice

Notions of probability

- Asset returns are subject to uncertainty
- Let S be the set of possible states of the world
- Roll of a fair dice: S={1,2,3,4,5,6}
- An *event* is a subset of S
- Ex: A={2,4,6} is the event that the roll is even
- A *probability distribution* is a function that assigns probabilities to each possible state of the word
- Ex: If dice is fair, P(s)=1/6 for all s ∈ {1,2,3,4,5,6}, and, for any event A:

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$$P(A)=\frac{\#A}{\#S}$$

Random variables

- A random variable X on S attaches a value to each possible state of the world
- Assets (risky strings of cash flows) are random variables
- Ex: X pays $1 of roll of dice is even, nothing otherwise:

$$P(X=1)=P(s \in \{2,4,6\})=0.5$$

Expectations

- The *expected value* of a random variable X is defined as:

$$E(X)=\Sigma_{s \in S} P(s)\ X(s)$$

- X pays $1 of roll of dice is even, nothing otherwise:

$$E(X)= P(s=1) \times 0 + P(s=2) \times 1 + P(s=3) \times 0$$
$$+ P(s=4) \times 1 + P(s=5) \times 0 + P(s=6) \times 1 = 0.5$$

Variances and standard deviations

- VAR(X) $= \Sigma_{s \in S} P(s) (X(s)-E(X))^2$
  $= E[X-E(X)]^2$

- X pays $1 of roll of dice is even, nothing otherwise:

VAR(X)=

$P(s=1) \times (0-0.5)^2 + P(s=2) \times (1-0.5)^2 + P(s=3) \times (0-0.5)^2$
$+ P(s=4) \times (1-0.5)^2 + P(s=5) \times (0-0.5)^2 + P(s=6) \times (1-0.5)^2$
$= 0.25$

- The standard deviation of X is the square root of its variance:

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- The standard deviation of X is the square root of its variance:

$$\sigma_X = \sqrt{VAR(X)}$$

Risk

- A random variable X is *risk-free* if VAR(X)=0 ⇔ X(s)=x for all s ∈ S
- It is *risky* if VAR(X)>0
- The closest asset we have to risk-free asset in the US (the world?) is a T-bill
- Yes, even today, S&P's nonsense notwithstanding

Covariance

- We need a notion of how two random variables X and Y are related:

$$\text{COV}(X,Y) = \Sigma_{s \in S} P(s)\ (X(s)-E(X))(Y(s)-E(Y))$$
$$= E[(X-E(X))(Y-E(Y))]$$

- COV(X,Y)>0 means that X tends to be high when Y tends to be high, and vice-versa

- Note 1: if X is risk-free, then COV(X,Y)=0
- Note 2: COV(X,X)=VAR(X)
- Note 3: COV(X,Y)=COV(Y,X)

Example

- X pays \$1 if roll of dice is even, Y pays \$1 if roll of dice is 4 or more

- Then E(X)=E(Y)=0.5, and:

$$
\begin{aligned}
COV(X,Y) = &\ P(s=1)(0-0.5)(0-0.5)+ P(s=2)(1-0.5)(0-0.5)+ \\
&\ P(s=3)(0-0.5)(0-0.5)+ P(s=4)(1-0.5)(1-0.5)+ \\
&\ P(s=5)(0-0.5)(1-0.5)+ P(s=6)(1-0.5)(1-0.5) \\
=&\ 1/12
\end{aligned}
$$

Coefficient of correlation

- $\rho_{X,Y}$=COV(X,Y) /($\sigma_X\sigma_Y$)
- Varies from -1 to 1
- $\rho_{X,Y}$=1 means that Y=a X +b, where a>0
- $\rho_{X,Y}$=-1 means that Y=a X +b, where a<0

Example

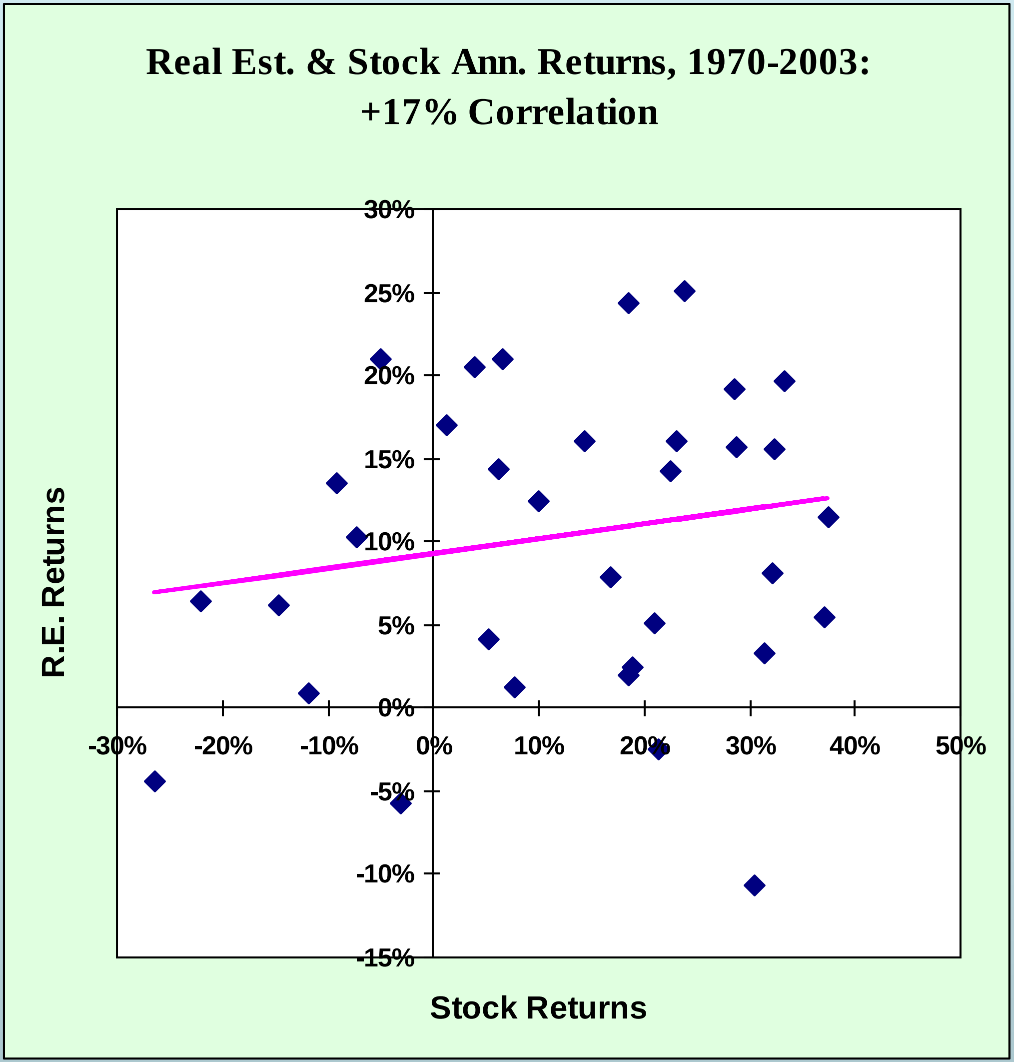
- X pays $1 of roll of dice is even, Y pays $1 if roll of dice is 4 or more

- $\rho_{X,Y} = COV(X,Y) / (\sigma_X \sigma_Y) =$

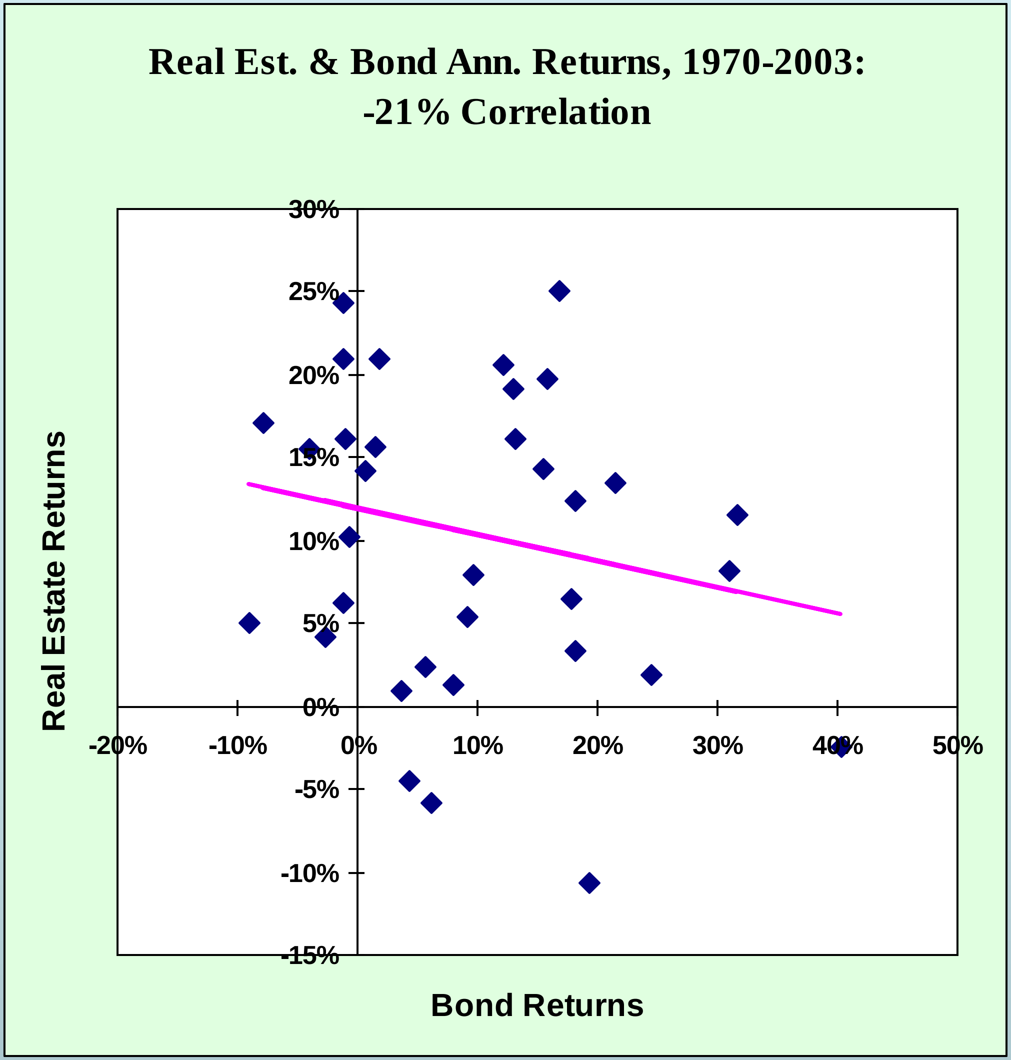
Example

- X pays \$1 of roll of dice is even, Y pays \$1 if roll of dice is 4 or more
- $\rho_{X,Y}$ =COV(X,Y) /($\sigma_X \sigma_Y$) = $\dfrac{\frac{1}{12}}{\sqrt{0.25 \times 0.25}} = \dfrac{1}{3}$

Real estate and stock returns



Real estate and bond returns



Mixing assets

- Let a and b be numbers, and X and Y be the returns on two assets
- Investing a in X and b in Y returns aX(s) + bY(s) in state s
- (a,b), in this context, is called a *portfolio*
- We write aX+ bY for the resulting random variable

Big facts

- $E(aX+bY) = aE(X) + bE(Y)$
- $VAR(aX)=a^2VAR(X) \Leftrightarrow \sigma_{aX} = a\ \sigma_X$
- $VAR(aX+bY)= a^2VAR(X) + b^2VAR(\mathbf{Y}) +2ab\ COV(X,Y)$
- $VAR(0.5X+0.5Y) =$

  $0.25VAR(X) + 0.25VAR(Y) +0.5\ COV(X,Y)$

Diversification

- Combining **risky** assets reduces risk unless $\rho_{X,Y}=1$
- Returns on assets that do not covary perfectly tend to offset each other, at least a little bit
- If they co-vary negatively, diversification is even greater
- If you bet the same amount on both red and black at the roulette, you're taking on virtually no risk

More facts

- $COV(aX+bY,Z) = aCOV(X,Z) + bCOV(Y,Z)$

More facts

- COV(aX+bY,Z) = aCOV(X,Z) + bCOV(Y,Z)
- And the big monster:

$$VAR\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j COV(X_i, X_j)$$

Financial economies

- Two dates: t=0, t=1
- Time in between is called the holding period
- N assets, available in fixed (given) supply
- Asset $i \in \{1,2,\ldots,N\}$ has random payoff X_i at date t=1
- If it costs q_i at date 0, return is $r_i(s)=X_i(s)/q_i-1$
- Expected return is $E(r_i)=E(X_i)/q_i-1$

Investors

- J investors, with given wealth to invest at date 0
- Choose a portfolio $(\alpha_1, \alpha_2, \ldots \alpha_n)$ where $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$
- α_i is the fraction of her wealth the investor spends on asset i
- If investor has wealth w and buys $(\alpha_1, \alpha_2, \ldots \alpha_n)$, she spends $\alpha_i w$ on asset i
- Note: α's can be negative $\Rightarrow$ *short-selling*

Portfolio risk and return

- Return on portfolio: $\Sigma_i \alpha_i r_i$
- Expected return: $E(\Sigma_i \alpha_i r_i) = \Sigma_i \alpha_i E(r_i)$
- Variance: $VAR(\Sigma_i \alpha_i r_i) = \Sigma_i \Sigma_j \alpha_i \alpha_j COV(r_{i,} r_i)$

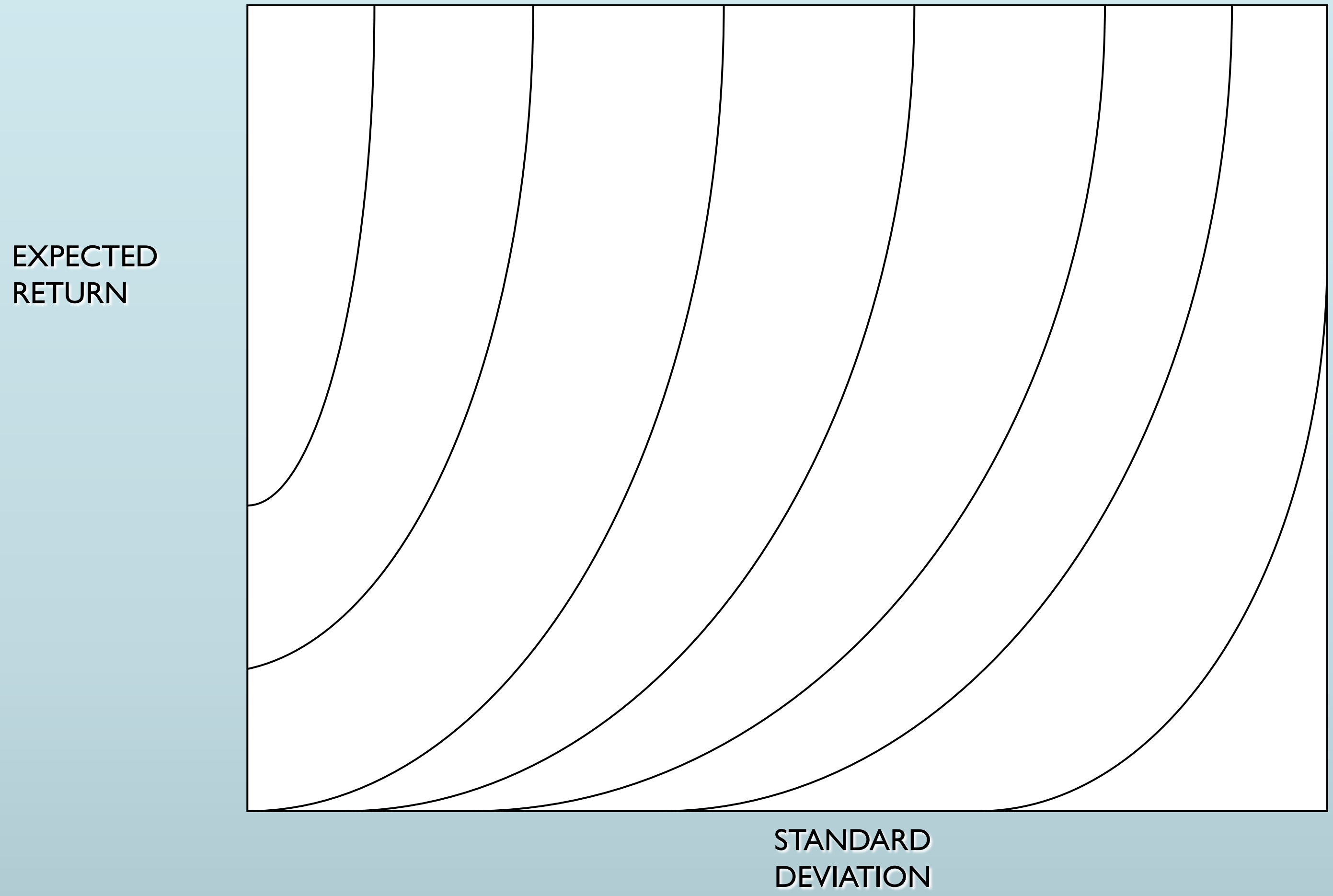
Mean-variance preferences

- Investors care about average (or mean) returns and standard-deviations (or variances)
- Holding variance the same, all investors prefer higher returns
- A *risk-neutral* investor only cares about expected returns
- A *risk-averse* investor prefers less risk, holding expected return the same
- A *risk-loving* investor prefers more risk, holding expected return the same

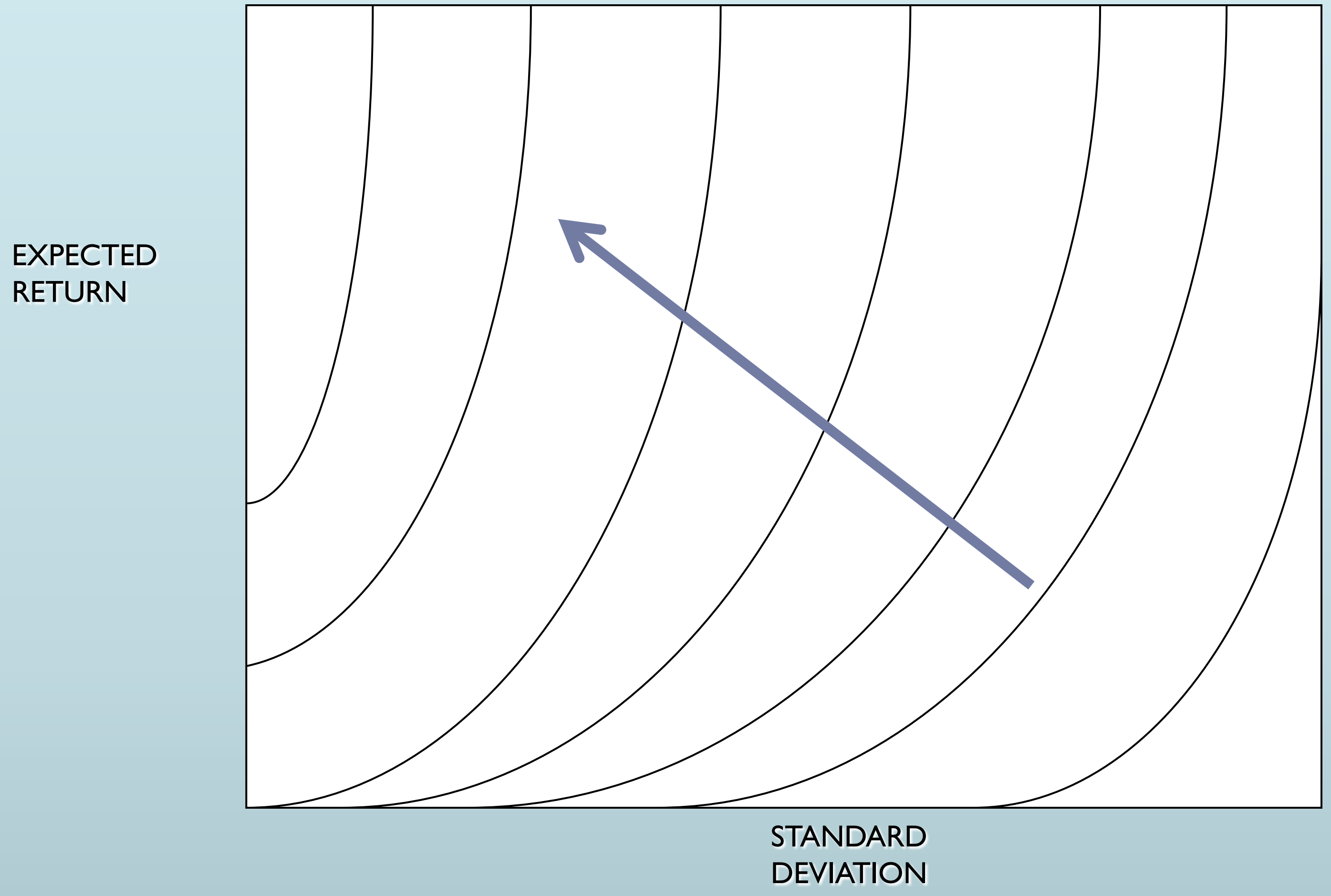
Indifference curves

- In the expected return/standard deviation plane, each risk/return combination gives the investor a given utility level
- Indifference curves connect risk/return combinations that give the investor the same utility level

Indifference curves of risk-averse investor



Utility rises as we move to the north-west



Equilibrium

- An equilibrium is a set $(q_1, q_2, \ldots q_n)$ of asset prices and a set of portfolio choices by all investors such that:

1. All investors choose the portfolio that maximizes their utility

2. Total demand for each asset equals supply

Law of one price

- The law of one price holds if whenever two portfolios yield the exact same payoff in all states, they cost the same.

- Remark: If there are no restriction on short-selling, the law of one price must hold in equilibrium

- Proof: take two portfolio with the same payoff but different prices. Buy the cheap one, sell the expensive one, no payoff implication at date 1, but you are richer at date 0.

Arbitrage

- A strong arbitrage is a portfolio with a negative price today and a non-negative payoff in all states at date 1
- A deviation from the law of one price is a strong arbitrage opportunity
- No strong arbitrage can exist in equilibrium

Fundamental theorem of finance

No arbitrage

$$\Longleftrightarrow$$

$$q_i = E^*(X_i) /(1+r) \text{ for all } i$$

where the expectation* is with respect to a synthetic probability distribution called the risk-neutral probability and r is the risk free rate

Most of modern finance prices assets by estimating the RNP first and then pricing assets as if agents were risk neutral

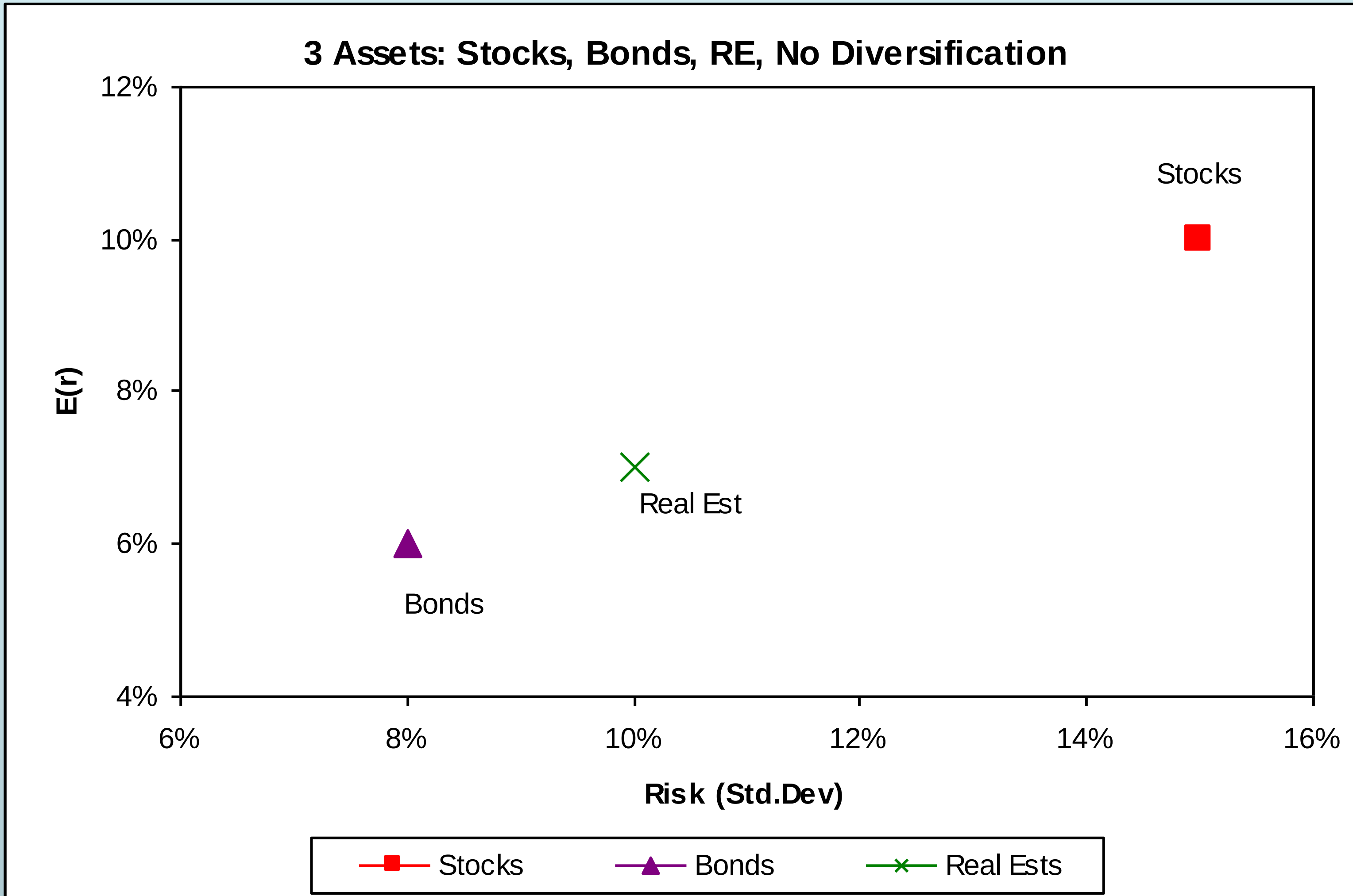
Classical portfolio theory

- All investors have mean-variance preferences, and are risk-averse
- Can divide their wealth across assets however they wish
- No taxes or transaction costs
- Investors have all the information they need about assets
- There is a risk-free asset, and investors can borrow and lend at will at the risk-free rate

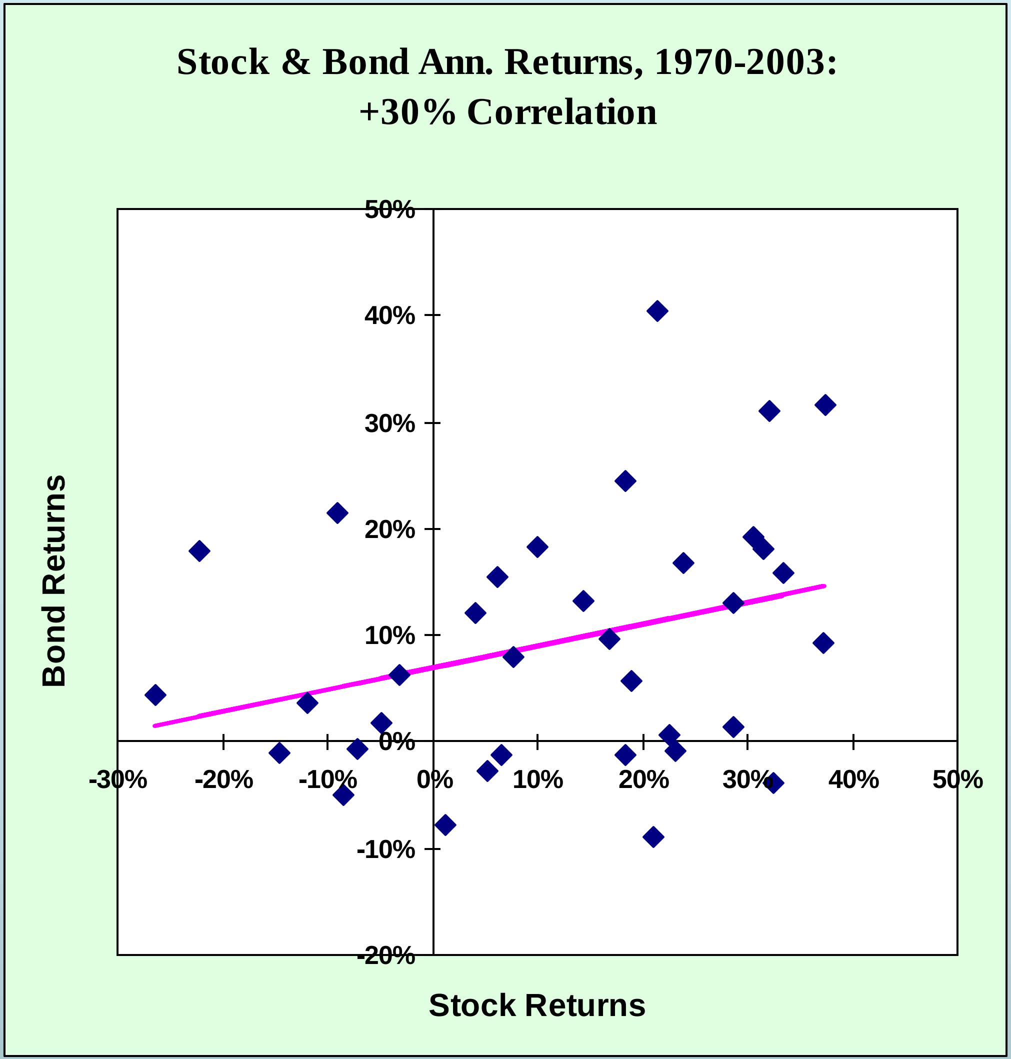
Feasible set

- Set of mean return/standard deviation investors can achieve
- Each possible portfolio is a point in the feasible set
- If there are at least 3 securities, feasible set is a mass with **no** holes
- If there is no risk-free asset, north-west boundary bends outward
- If there is a risk-free asset, north-west boundary is a straight line

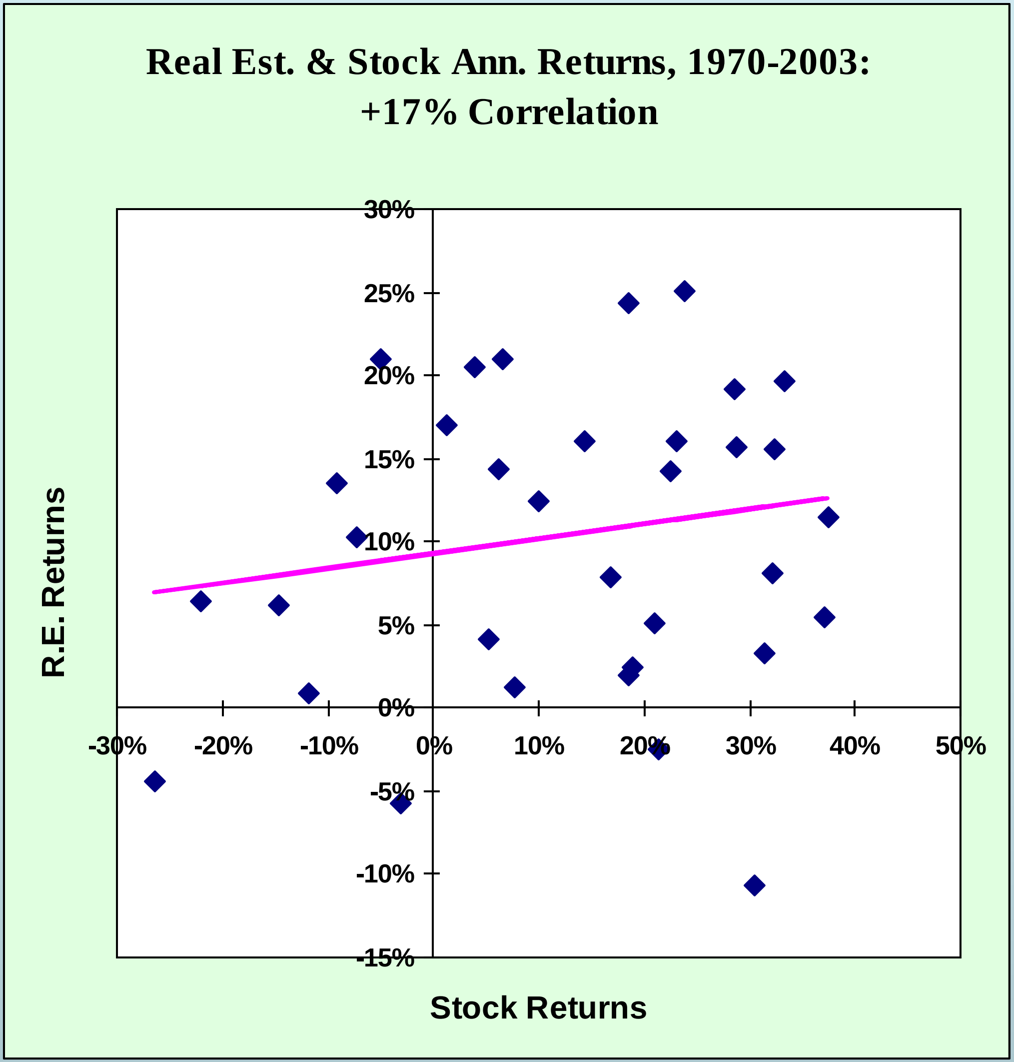
An example with 3 assets



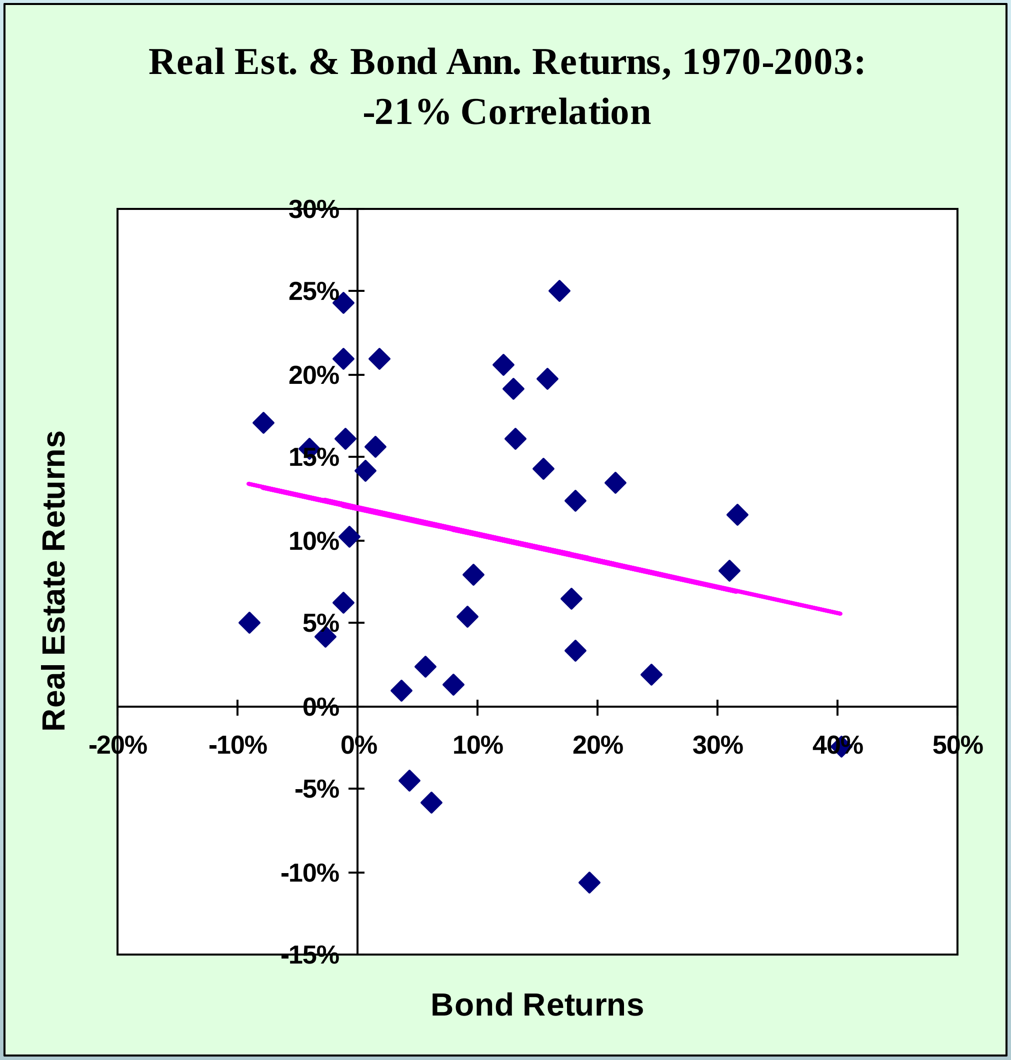
Bond and stock returns



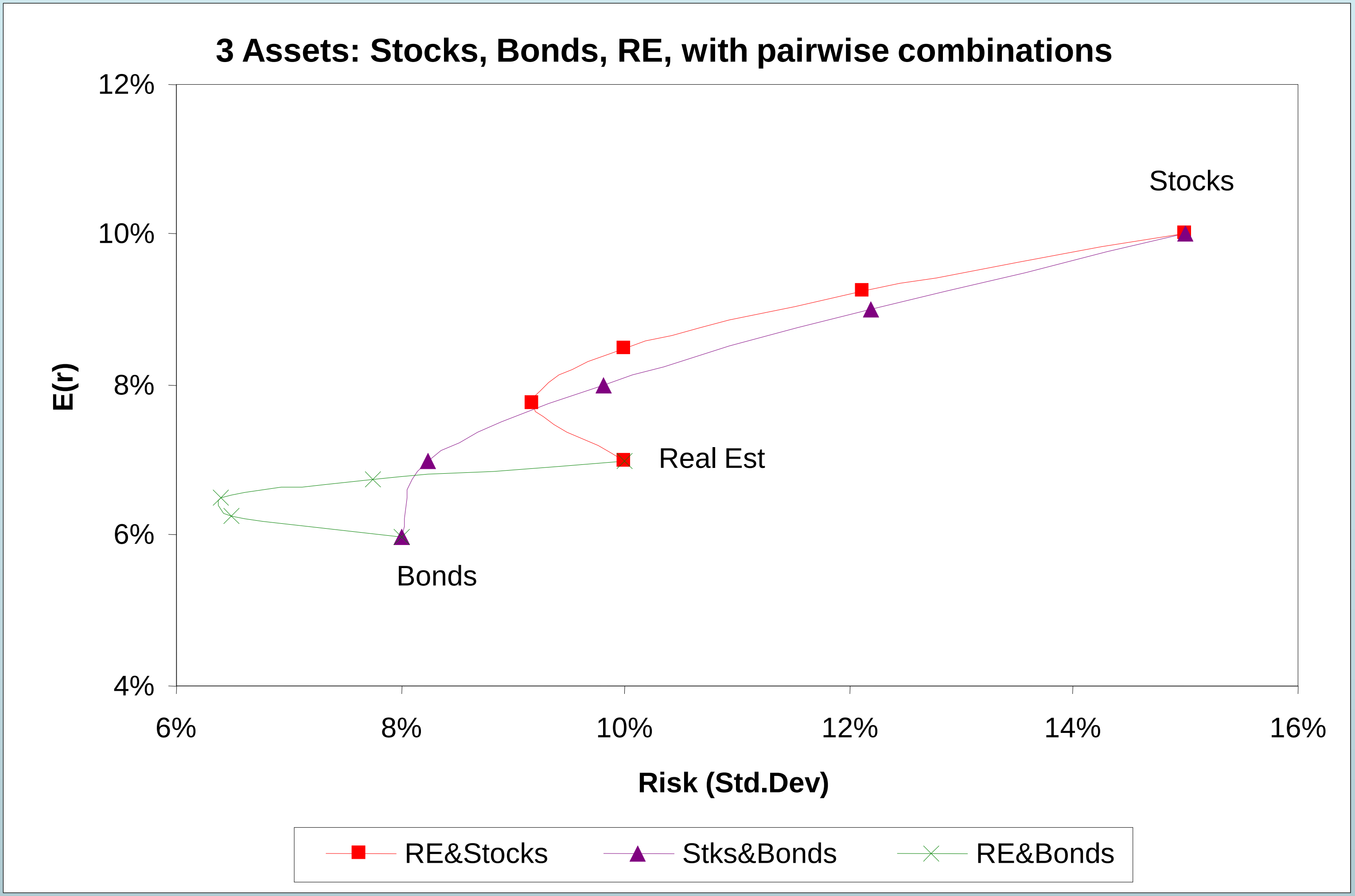
Real estate and stock returns



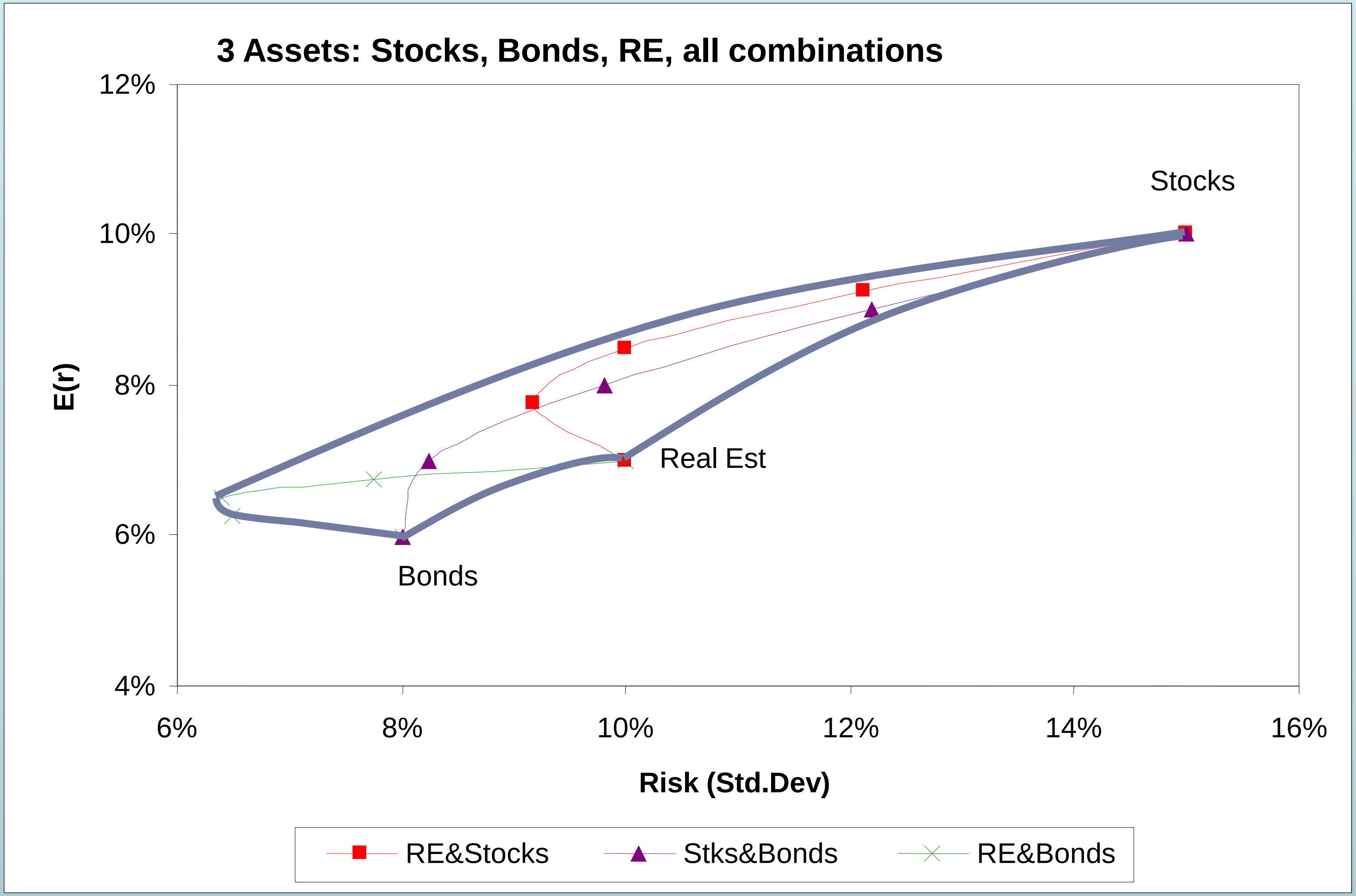
Real estate and bond returns



Combining pairs of assets



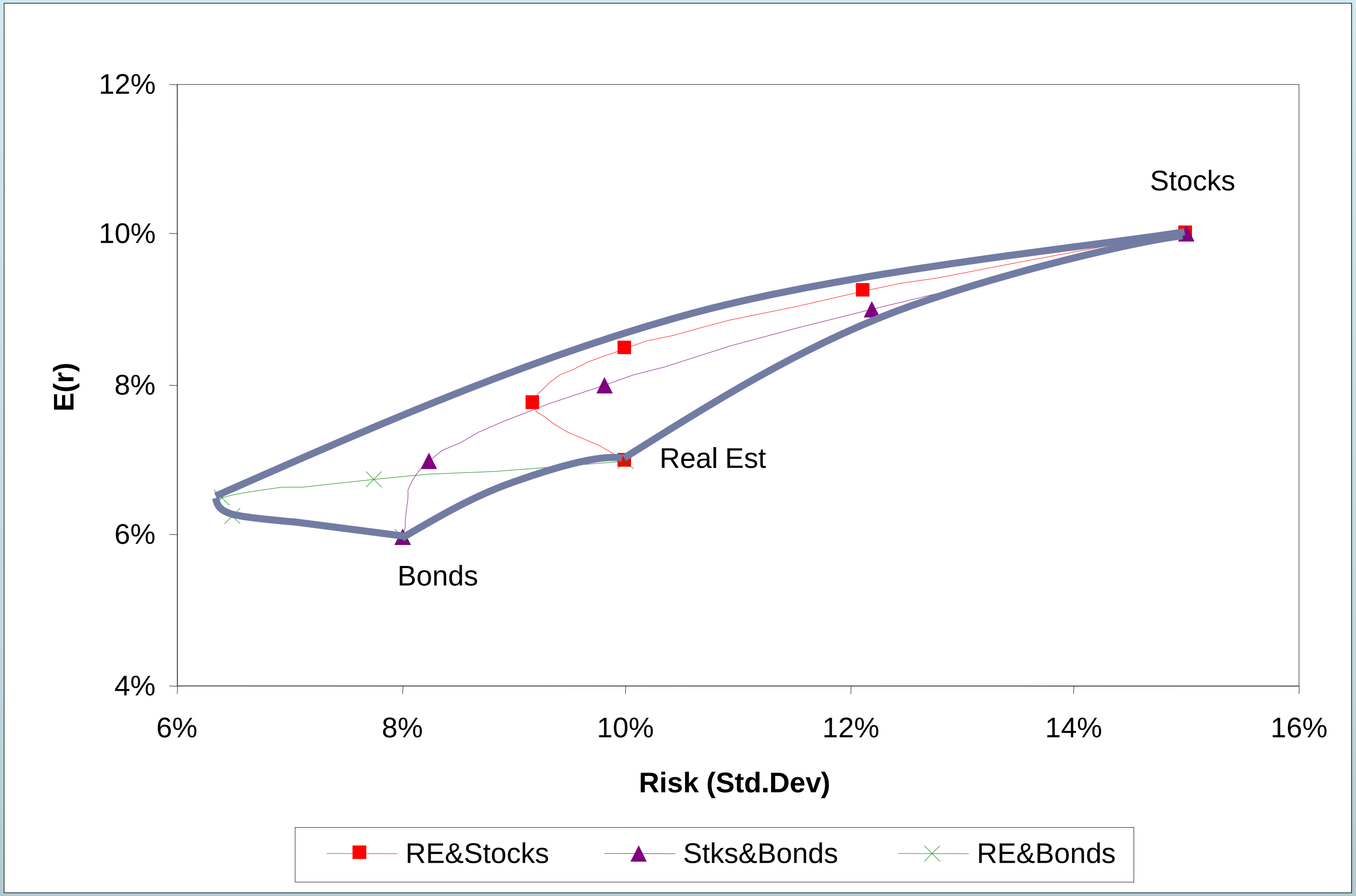
Combining all assets: feasible set



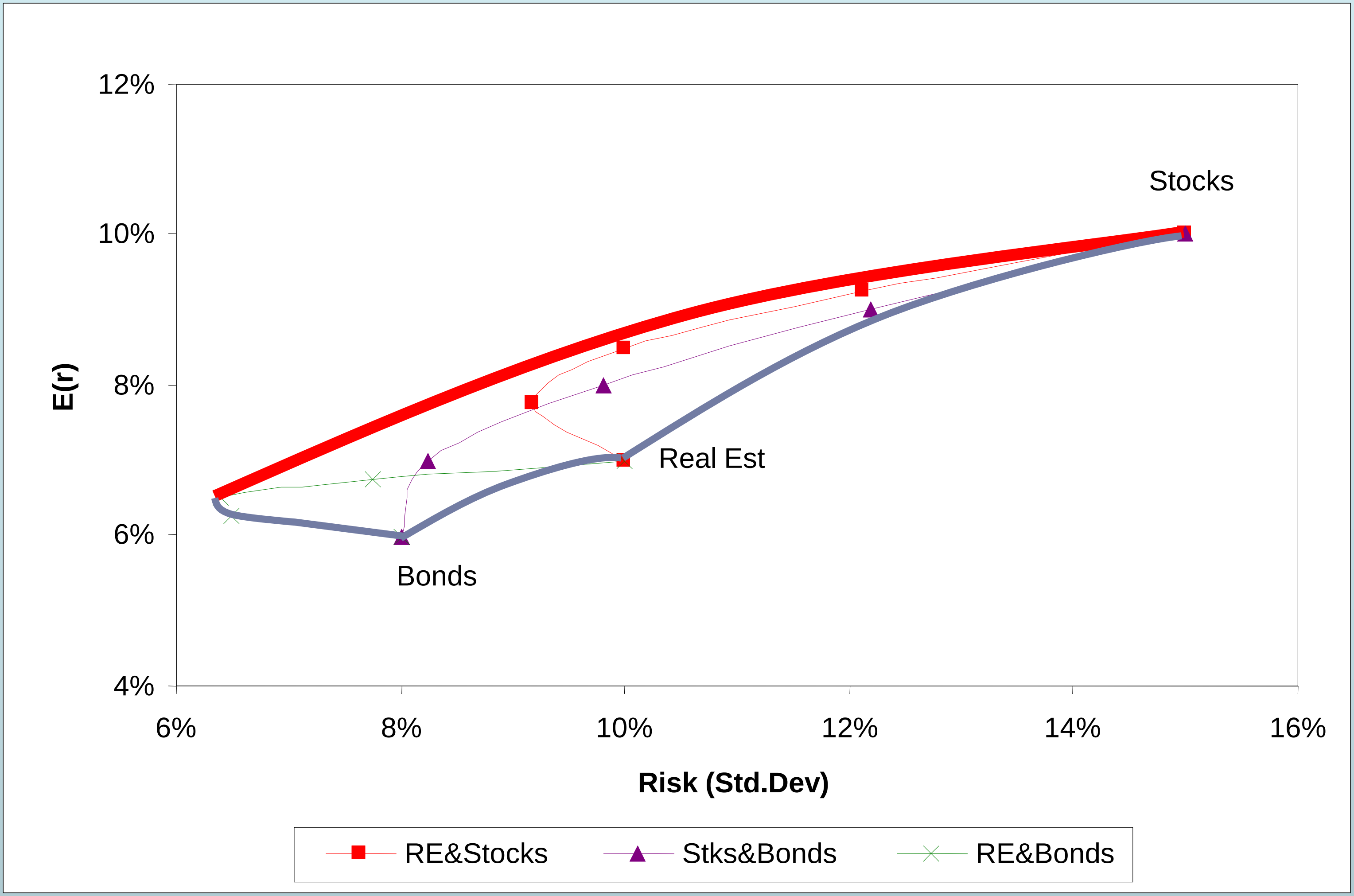
Efficient set

- The north-west boundary of the feasible set is called the *efficient set*
- Portfolios in the efficient set are called *efficient portfolios*
- In equilibrium, all investors hold portfolios that are on the efficient set

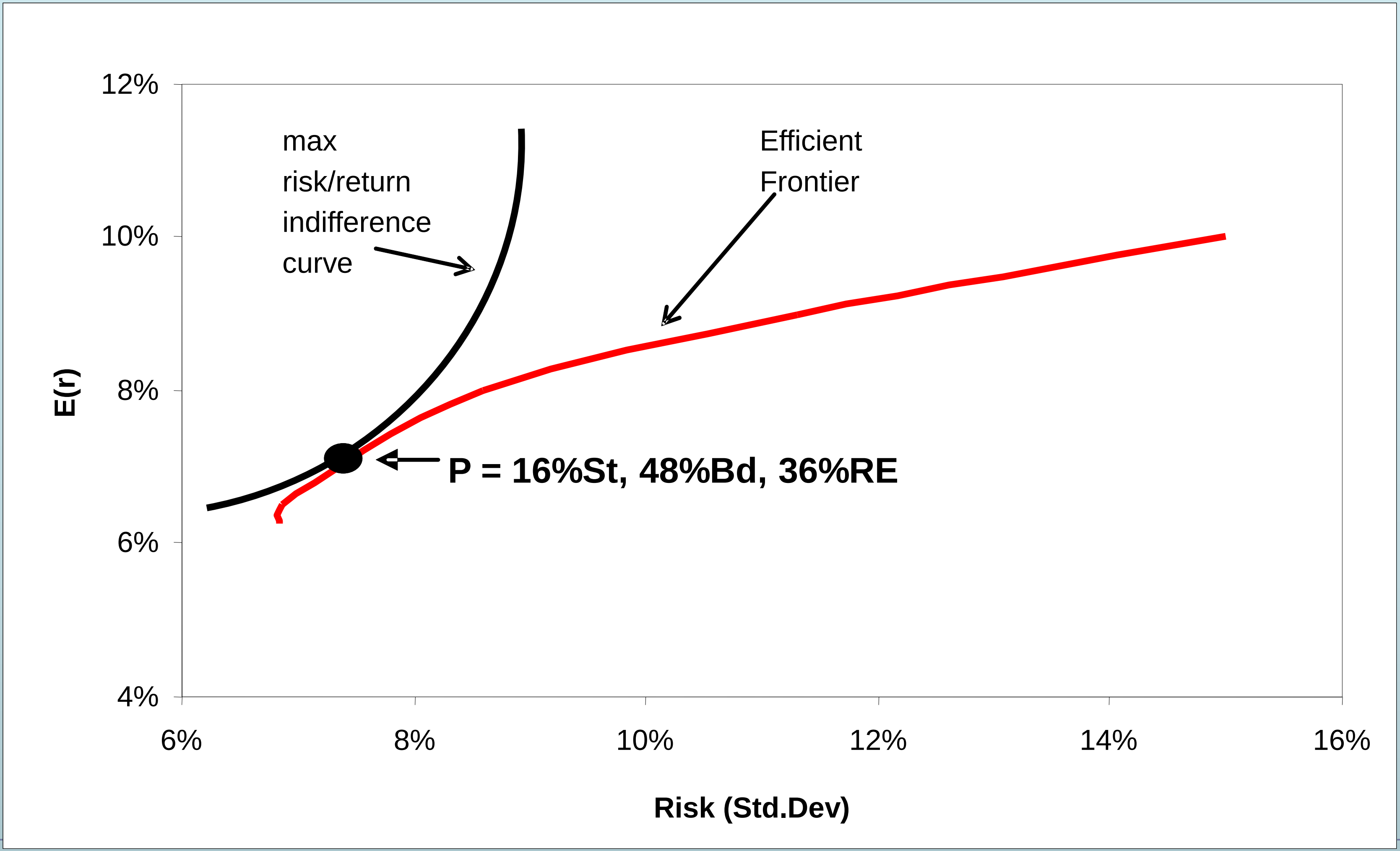
Efficient set



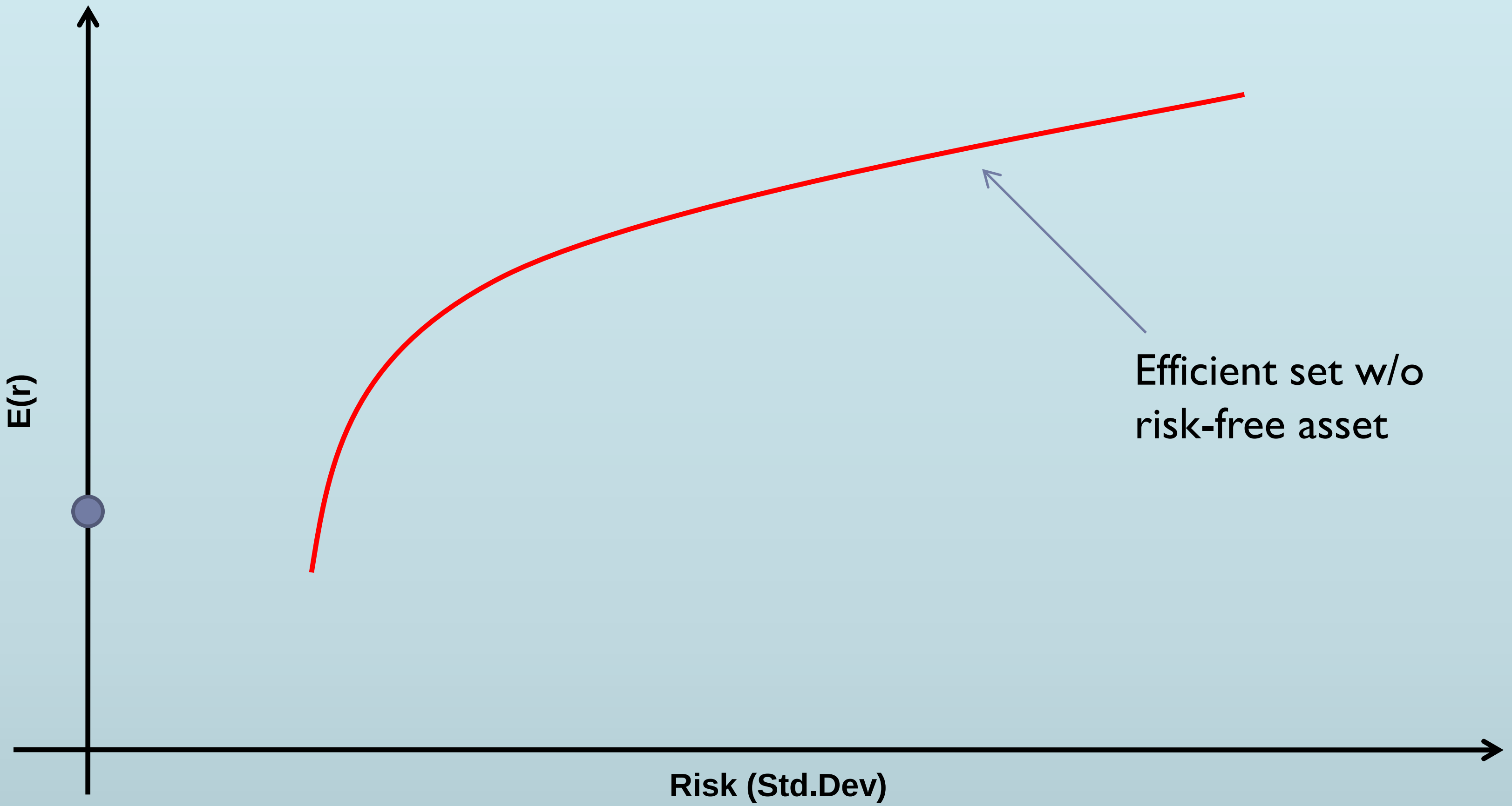
Efficient set

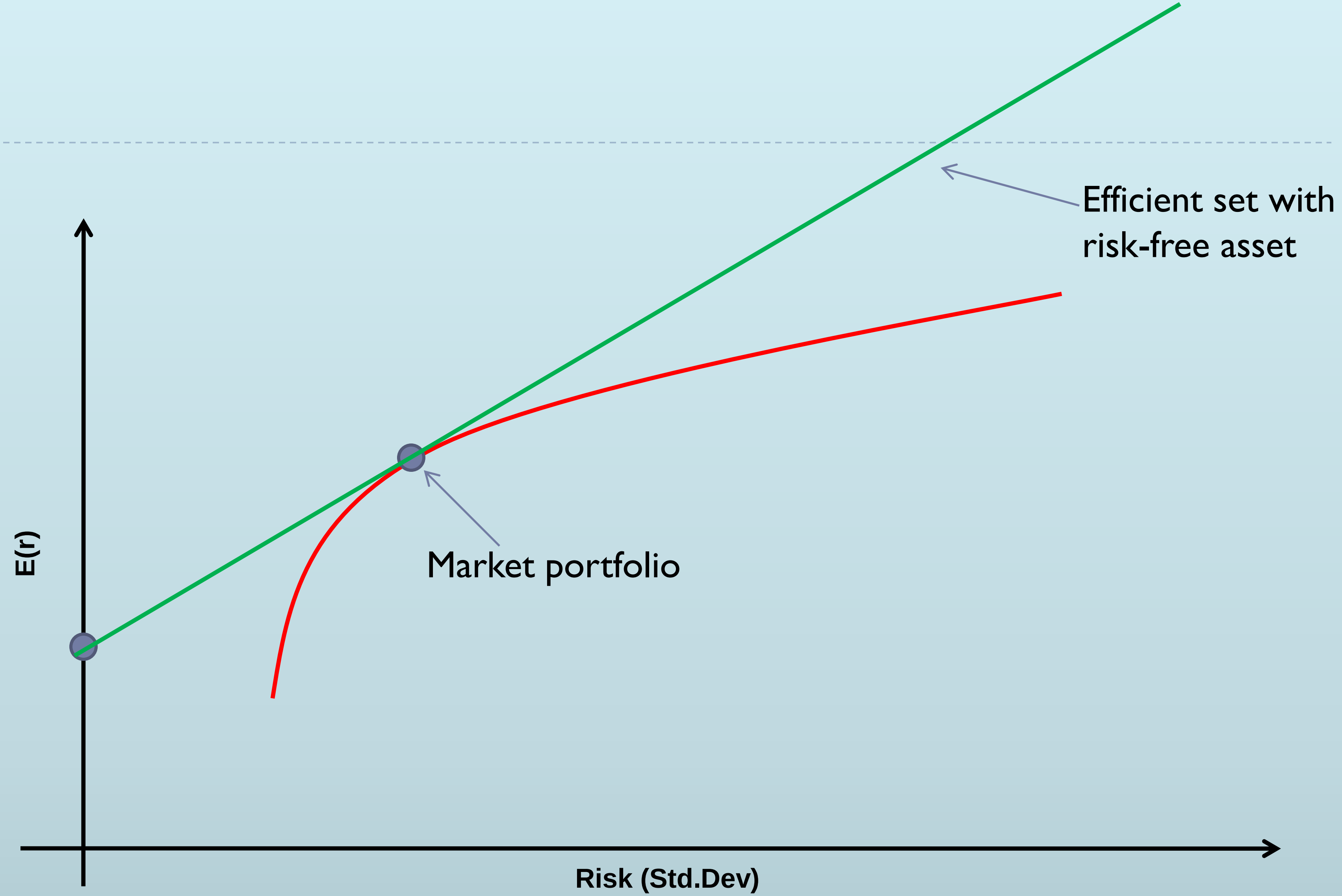


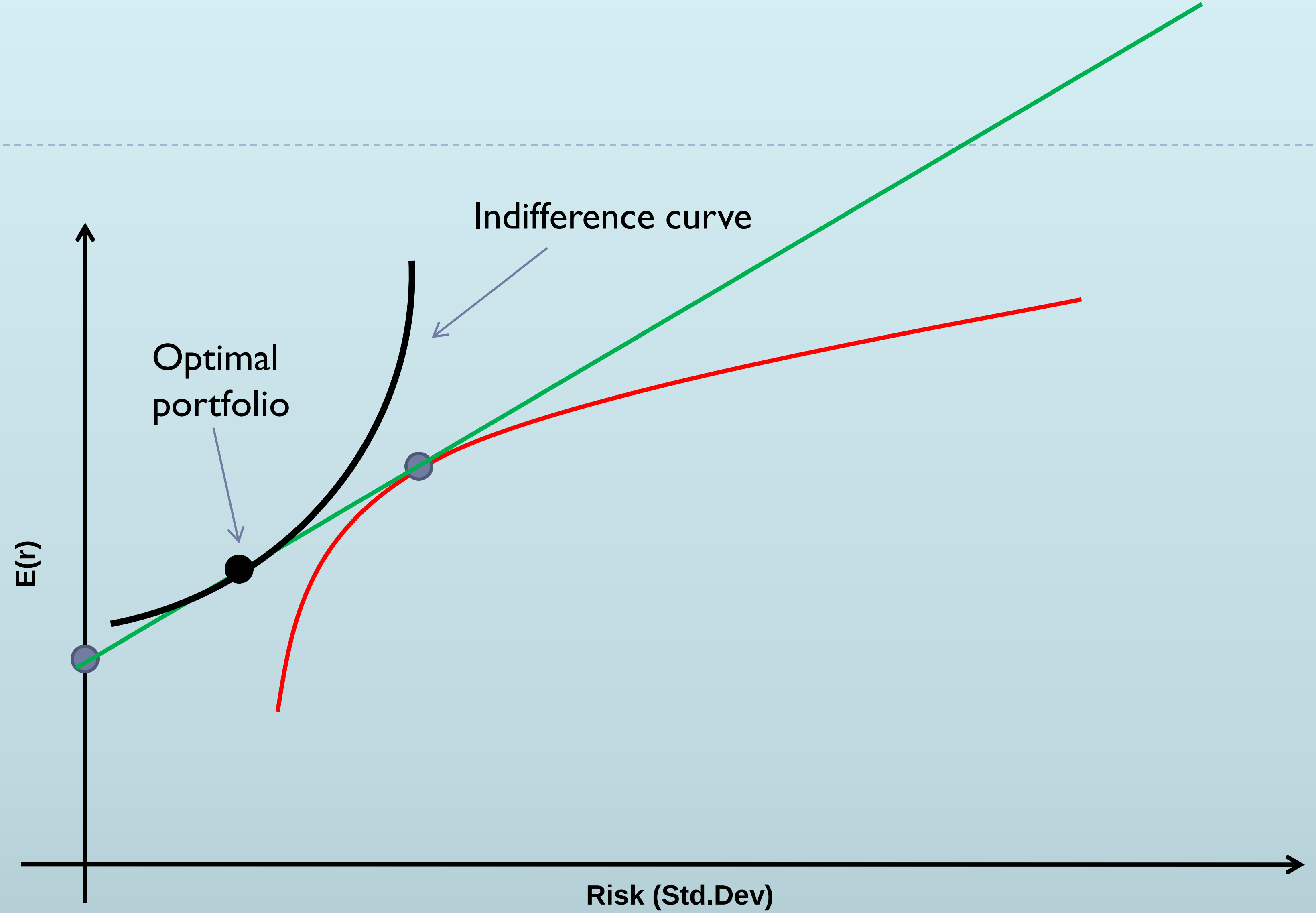
Optimal portfolio for a risk averse investor



Adding a risk-free asset




Efficient set with
risk-free asset
E(r)
Market portfolio
Risk (Std.Dev)


Indifference curve
Optimal
portfolio
E(r)
Risk (Std.Dev)

Two-portfolio theorem

- With risk-free asset, efficient set begins at portfolio that puts all wealth in risk-free asset, and touch the risky part of the feasible set in exactly one point
- That point is called the market portfolio
- **Theorem**: In equilibrium, all investors hold a portfolio made of a positive investment in the market portfolio, and a positive or negative investment in the risk-free asset

Market portfolio

- All risky assets have positive weight in it
- The risky-part of all investors portfolios is the same, namely the market portfolio
- It follows that the market portfolio can be computed as the fraction of total risky holdings in a given asset
- Decent practical proxy: capitalization-weighted index, such as the S&P500

Capital Asset Pricing Model (CAPM)

- What should be the average return on asset i in equilibrium? Equivalently, what should be its price?
- Intuitively, riskier assets should command a higher return
- Investors should be compensated for the risk a given asset contributes to their portfolio
- This contribution depends on how it co-varies with all elements of the portfolio, including itself

Capital Asset Pricing Model (CAPM)

- Theorem:

$$E(r_i) = r_f + \beta_i \, [E(r_m) - r_f]$$

where:

Capital Asset Pricing Model (CAPM)

- Theorem:

$$E(r_i) = r_f + \beta_i\,[E(r_m) - r_f]$$

where:

$$\beta_i = \frac{COV(r_i, r_m)}{VAR(r_m)}$$

CAPM

- Investors want to be compensated for a very specific form of risk: the asset's beta

- Return on a given asset is the risk-free rate plus a risk premium

- Risk premium is the product of beta (the *quantity of risk*) and $E(r_m) - r_f$ (the *market price of risk*)

CAPM in real estate

- Say you are considering a property, and have a forecast of associated cash flows
- How should we discount those cash flows?
- *All* we need is similar property's beta
- Two problems: r_m and r_i
- For the first, S&P500 is fine, maybe best
- For the second one, one can use REIT data (see homework)
- Apparent problem: REITs are bundles of properties, rather than single properties. This reduces risk, right?
- Right, but irrelevant
- True problem: Liquidity corrections have to be made

Diversifiable risk does not matter

- Asset i's beta is the slope you get if you regress r_i on r_m
- Therefore, $r_i = r_f + \beta_i (r_m - r_f) + \varepsilon_i$
  where: $COV(r_m, \varepsilon_i) = 0$
- It follows that $VAR(r_i) = \beta_i^2 VAR(r_m) + VAR(\varepsilon_i)$
- Asset's risk is the sum of its *systematic risk*, and its *specific (unique, diversifiable) risk*
- Only the first type of risk affects pricing

The REIT approach

- Looking at bundles of properties rather than single properties to estimate a given project's beta is just fine
- REITs however are much more liquid assets than single properties
- This is reflected in returns
- A *(lack of) liquidity correction* should be added to required rate on single property
- Guidance can be found in the private vs. public literature
- Even more fudgy: often people impose "lack of comparability" premia on discount rates in recognition that no two properties are alike

Investment value vs. market value

- Lack of comparability also stems from the fact that a particular investor may be able to squeeze more value out of property than other investors could

- Value to a given investor is called the *investment value*

- Can exceed *market value*, the value at which property would sell in competitive markets because of:
  1. Private information
  2. Skill
  3. Investor specific criteria: preferences, taxes…

The REIT approach - summary

1. Find a set of REITs who invest in the right property type (location, purpose…)

2. Get their betas, and average them: β_i

3. Estimate/guess r_f and $E(r_m)$ for the relevant holding period

4. Invoke CAPM: $E(r_i) = r_f + \beta_i [E(r_m) - r_f]$

Issues

- Liquidity correction: it is estimated that REIT-held assets embed a 12-22% liquidity premium over directly held assets
- Another solution: use right subcomponent of an index such as NCREIF instead of REIT data
- Disadvantages: premium properties only, less freedom to tailor comparables
- CAPM does not work well, and no FAMA-FRENCH correcting factors have been proposed for real estate
- Leverage matters, more on that in a few slides

A cute CAPM point

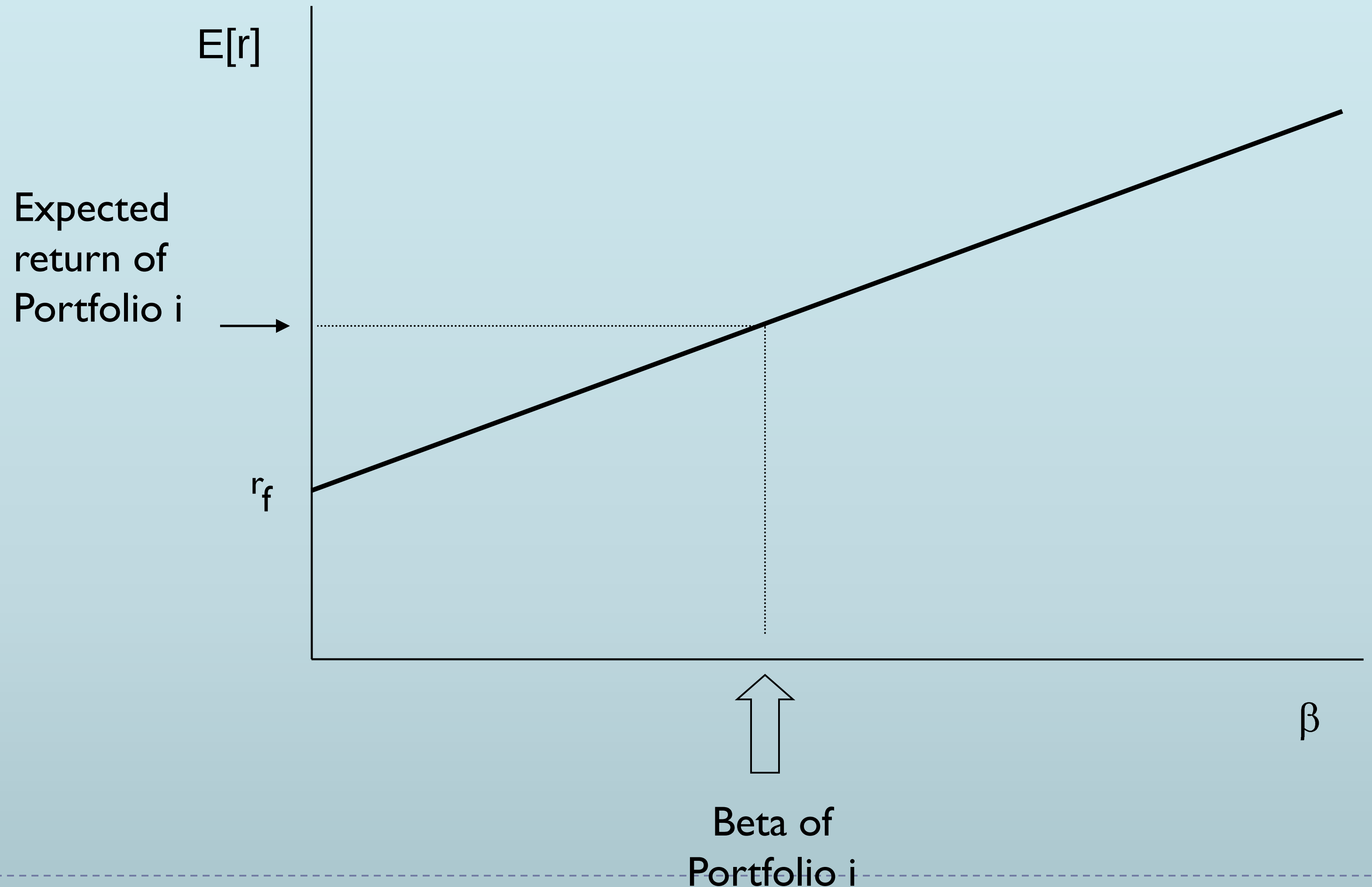
- A risky asset can in principle earn less than the risk-free asset
- All you need is an asset that co-varies negatively with the market portfolio
- Probably hard to find, but a theoretical possibility

A key CAPM point

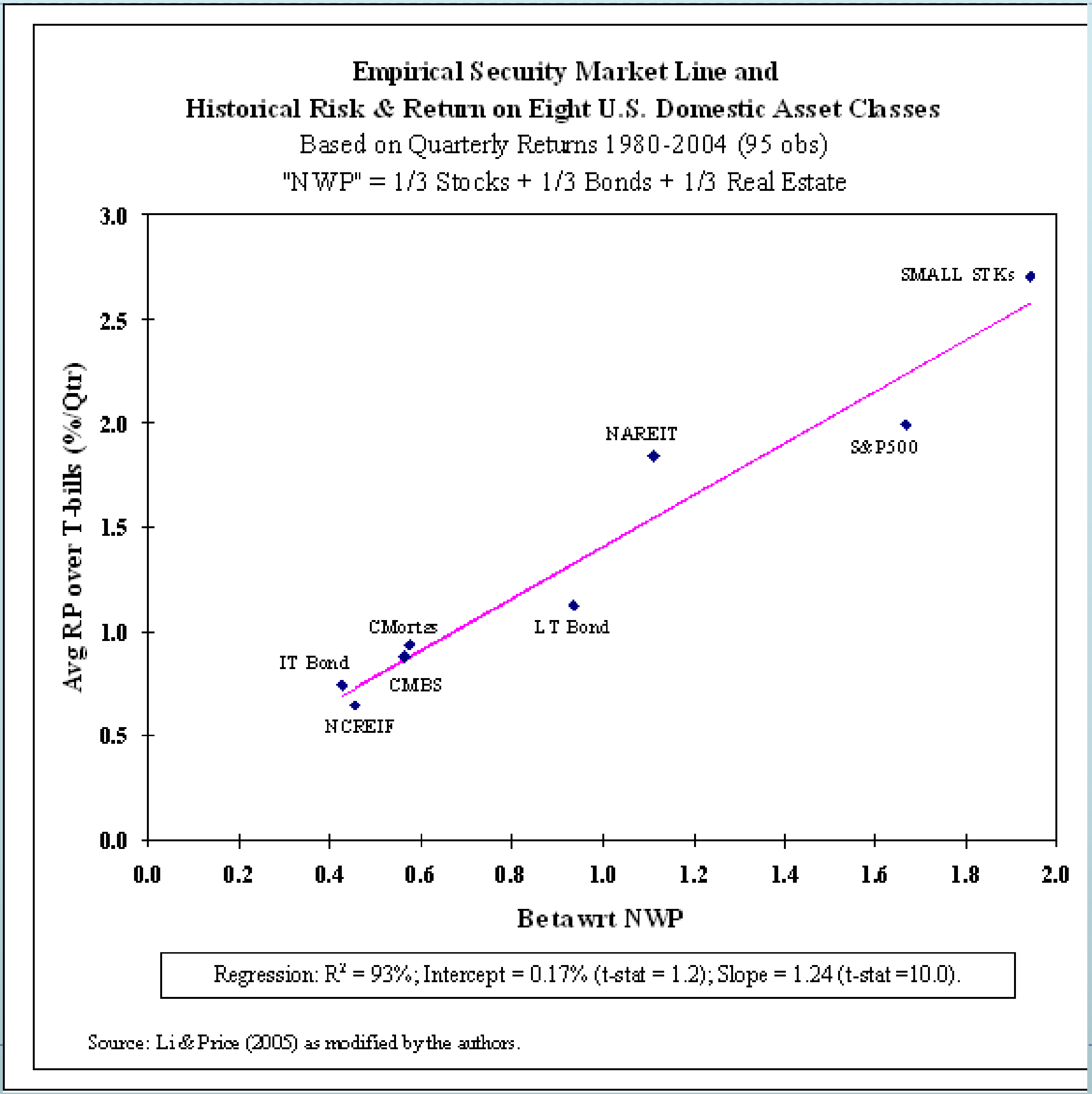
- β's are linear
- Consider a portfolio made of share α_1 in asset 1 and α_2 in asset 2
- The portfolio's beta is:

$$\begin{aligned}\beta_P &= COV(\alpha_1 r_1 + \alpha_2 r_2, r_m)/\mathbf{VAR(r_m)} \\ &= [\alpha_1 COV(r_1, r_m) + \alpha_2 COV(r_2, r_m)]/\ \mathbf{VAR(r_m)} \\ &= \alpha_1 \beta_1 + \alpha_2 \beta_2\end{aligned}$$

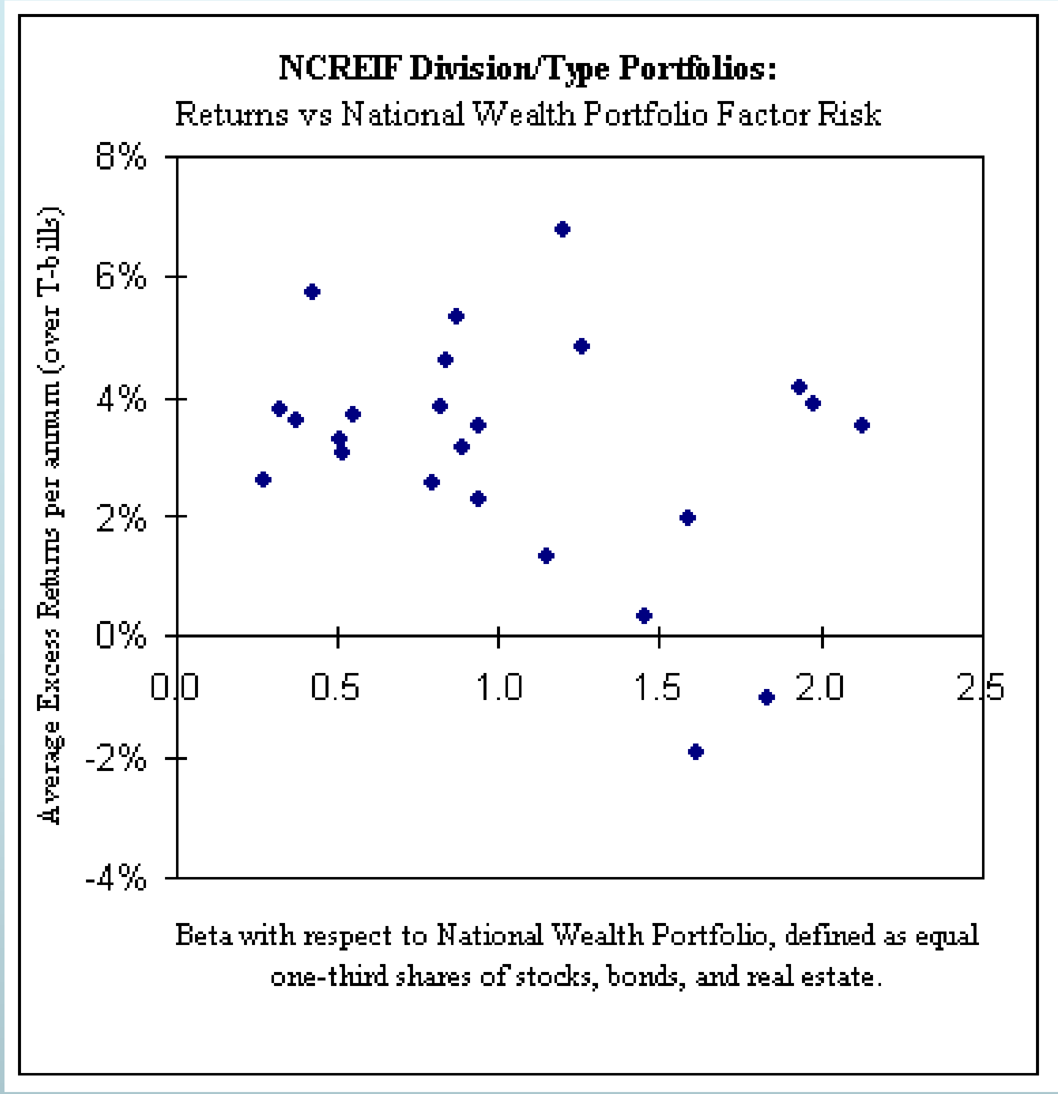
The CAPM bottom line



CAPM works OK for broad asset classes

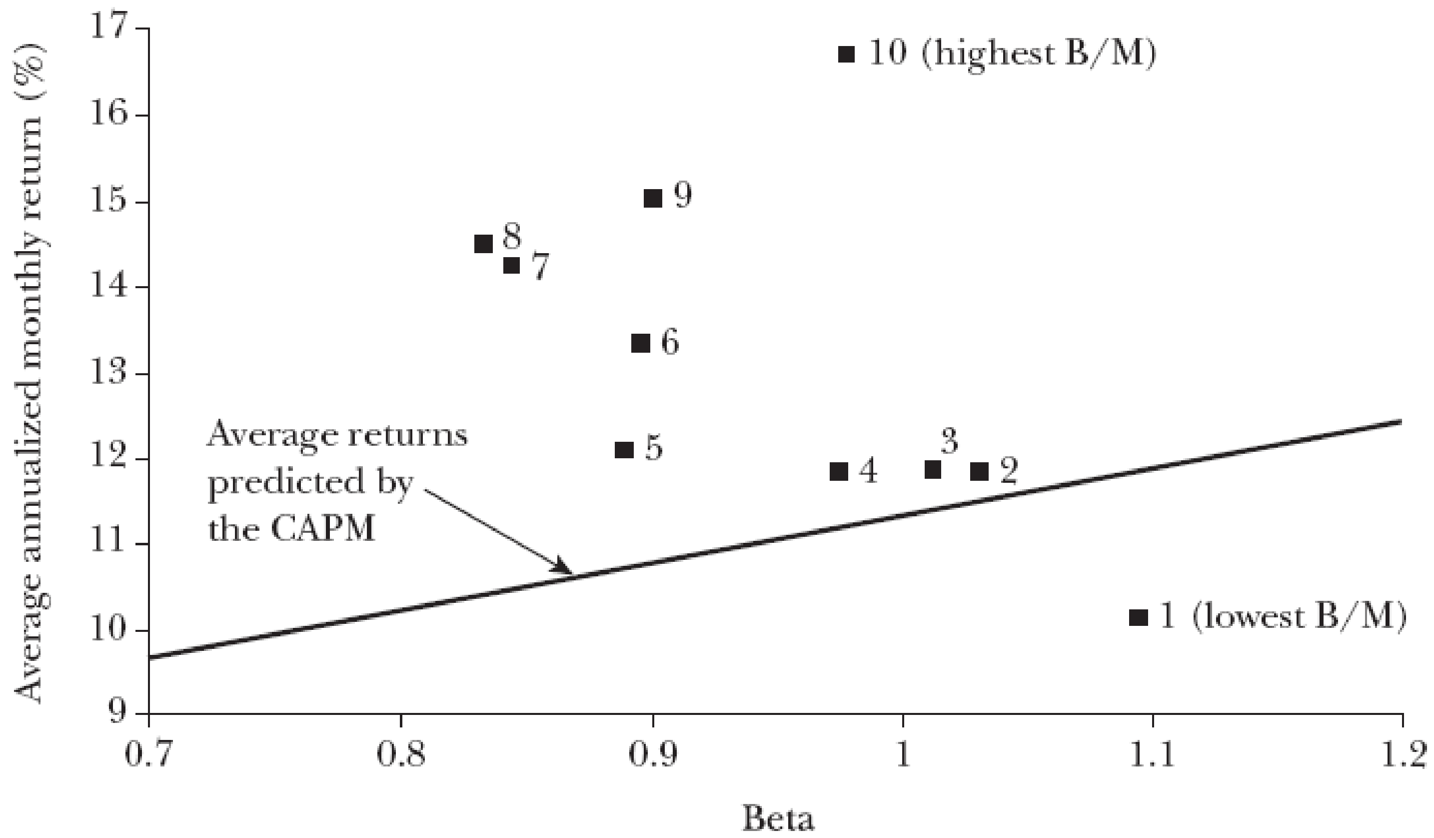


Not so well for narrower classes



True outside of real estate as well

Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on B/M, 1963–2003



Option pricing

- An *option* is a contract where one party grants (sells) the other party the right to buy or sell an asset at a specific price, within a specific time period.

- A *European option* can be exercised only at the expiration date

- An *American option* can be exercised at any point before the expiration date

- We know a lot about how to price the first type, much less about how to price the second

Option terminology

- A *call* option is the right to **buy**, a *put* option is the right to **sell**
- The price at which the option buyer may buy or sell is called the *strike* or *exercise price*
- A call option is *in the money* if the asset price rises above the strike price
- A put option is in the money in the opposite situation

Binomial option pricing

- Two possible states at date 1: up and down, with probability p and (1-p)
- Underlying asset pays *u* if up state is realized, *d*<*u* otherwise
- Price of asset at date 0 is q
- Consider a call option on this asset with strike price d<s<u
- What should be the call option's price?

Binomial option pricing formula

- Option pays u-s in up state, nothing in down state
- A strategy that gives the exact same payoff is
  1. buy quantity (u-s)/(u-d) of the asset at date 0,
  2. borrow (u-s)d / [(u-d)(1+r_f)] at risk free rate
- Arbitrage says that the call option and this strategy must pay the same. This gives:
- Call option price=

Binomial option pricing formula

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- Arbitrage says that the call option and this strategy must pay the same. This gives:
- Call option price=(u-s)/(u-d) q - (u-s)d / [(u-d)(1+r_f)]

Can be generalized

- Many states
- In fact, implies the famous Black-Scholes formulae

Implications

- The higher the strike price, the **lower** the value **of a call option**

- The bigger (u-d), the higher the value

- Holds in full generality: the more volatile the underlying asset, the more valuable the option contract

Options in real estate

- Most real estate projects have option-like aspects: develop (call), expand (call), upgrade (call), contract (put), abandon (put)…
- Research suggests that these real options account for a significant part of property values

Modigliani-Miller (MM)

- Does capital structure matter?
- Does the value of an asset depend on the mix of debt and equity that is used to finance its purchase?
- No, at least absent taxes, transaction costs or limits, and other frictions
- Obvious from CAPM: asset value depends on its payoffs alone

Arbitrage argument

- Consider two properties with the same random payoff X over t=1,2,3, …
- First property is purchased with equity E and debt D, its value at date 0 is $V^L=E+D$
- We assume that property lives for ever, and keeps structure fixed
- L for *levered* or *leverage*
- Second property is 100% equity financed, and has value V^U
- Can we have $V^L>V^U$?

Two strategies

- Strategy 1: Buy fraction α of levered asset's equity, which costs αE

- Payoff: $\alpha(X - Dr_f)$

- Strategy 2: Borrow αD and buy αV^U of equity in unlevered firm, which costs:

$$\alpha V^U - \alpha D = \alpha(V^U - D) < \alpha(V^L - D) = \alpha E$$

- Payoff: $\alpha X - \alpha Dr_f$

- Violation of the law of one price

What does MM tell us?

- Not so much that capital structure does not matter

- It says that if CS matters, it must be because of the frictions MM assume away:
  1. Taxes
  2. Costs associated with financial distress
  3. Agency problems (manager incentives vs. shareholder objectives)
  4. …

Return on equity

- Unlevered case: $r^U = X / V^U$
- Levered case: $r^E = (X - r^f D) / E = r^U + (D/E)(r^U - r^f)$
- Leverage: more debt means more return on equity as long as $E(r^U) > r^f$
- What's the catch? Risk goes up:
- $VAR(r^E) = VAR(r^U)(1 + D/E)^2$

Levered betas

- How does the beta of the levered property's equity compare to the beta of the unlevered property?
- $\beta^L = \beta(r^E) = \beta(r^U + (D/E)(r^U - r^f))$
  $= (1 + (D/E))\,\beta^U$
- It is higher, confirming that leverage implies risk
- Some stake-holders (debt-holders) assume "no" risk leaving equity holders to bear more risk

Weighted average cost of capital (WACC)

- WACC= E/(E+D) $E(r^E)$ + D/(E+D) r^f
- MM proposition II: WACC= $E(r^U)$ regardless of D
- WACC fact: the asset's value is the expected present value of all future cash flows discounted at the WACC
- Loosely speaking, a positive NPV when discounted at WACC means that cash-flows, in expected terms, are sufficient to meet the expected returns of all stake-holders

When reality strikes: Taxes

- If asset's owner is a corporation, they face taxes, but debt payments are tax deductible

- Net cash flows in each period, are:

$$X - \tau(X - Dr^f) = (1-\tau)X + \tau Dr^f$$

- The last term is called the tax shield, it adds value to the asset

- One shows: $V^L = V^U + \tau D$

- General principle: APV=NPV(property)+ NPV(financing)

If debt’s so great, why use equity at all?

- MM abstract from issues associated with financial distress
- Distress is costly both for obvious reasons and more subtle ones
- As a result, optimal debt-to-value ratio is less than 100%

Other MM results with taxes

- Unlevered case: $r^U = X(1-\tau)/V^U$
- Levered case: $r^E = r^U + ((1-\tau) D/E)(r^U - r^f)$
- $\beta^L = (1+(1-\tau) D/E)\,\beta^U$
- $WACC = E/(E+D)\; E(r^E) + D/(E+D)\,(1-\tau)\, r^f$
- Discounting expected net-of-taxes cash flows at WACC continues to give the right asset value answer

The WACC method

1. Project after-tax cash flows: $X(1-\tau)$
2. Discount at WACC
3. Result: D + E

Practical implementation

- Cost of debt is “easy”
- Cost of equity is tough:

1. Find the beta of “similar” assets
2. Unlever those betas: $\beta^U=(1+(1-\tau)\ D/E)^{-1}\ \beta^L$, average
3. Relever using the actual financing mix used in project under study
4. Invoke CAPM

Method’s advantages

1. Works in some theoretical contexts
2. Has intuitive appeal
3. Time-tested
4. Industry standard
5. What’s better out there?

Method's drawbacks

1. Assumptions that make it OK don't hold in practice
2. Levered beta formulae very MM specific
3. Relies on CAPM's approximate validity
4. Often misapplied: one-size WACC don't fit all projects
5. For private projects, what's the market value of debt, what's the market value of equity?

Summary

1. Portfolio risk, diversification
2. CAPM
3. What options are, and what makes them valuable
4. Leverage
5. Financing can create or destroy value
6. WACC