Risk, return, and asset prices

Corporate Finance

Building blocks (one period)

- Consider an investment that pays $100 in one period and costs $90 today

- The return on that investment is:

$$r = \frac{(100-90)}{90} = \frac{100}{90} - 1 \approx 11.11\%$$

- I need to invest $90 to generate $100 a period from now assuming a return of 11.11%

- In finance parlance, $90 is the present value of $100 a period from now assuming a discount rate of 11.11%

Building blocks (multiple periods)

- Consider an investment that pays $\$50$ in one period and another $\$50$ in two periods and costs $\$90$ today

- One measure of the return on this investment is the solution to:

$$90 = \frac{50}{1+r} + \frac{50}{(1+r)^2}$$

which is roughly $r = 7.32\%$

- If I invest $\frac{50}{1+r}$ today for 1 year at 7.32% and then $\frac{50}{(1+r)^2}$ for 2 years at 7.32% I will replicate the project's cash flows

- Put another way, I can replicate the project's cash flow if I have a technology that give me 7.32% per period

- This measure of return is called the *yield to maturity (YTM)*

Risk

- When cash-flows are uncertain, we can measure the project's return based on *expected cash flows* rather than promised cash-flows

- This is the cash flows I'd get on average if I invested in a large number of projects (or a project a large number of times) with no correlation across attempts

- In that case, r is called the *internal rate of return (IRR)*

YTM vs IRR

- On a project with specified cash-flow promises, the YTM is the return you will get if all payments are made as planned
- No default or prepayment, in particular
- It is the IRR if all goes according to the plan
- The IRR is the return associated with expected cash flows, the cash flow one expects on average on a project of this type
- Typically,* $YTM < IRR$

Returns in practice: debt math

- A debt contract stipulates:
  1. An initial balance b_0
  2. A frequency of payments and total number T of payments (term to maturity)
  3. An interest rate r_t for each period $t = 1,2, \ldots T$
  4. Payments m_t for each period $t = 1,2, \ldots T$
- Debt algebra:
  1. At a given date, interest due is $b_{t-1}\, r_t$
  2. Balance at date t is $b_t = b_{t-1} + b_{t-1} r_t - m_t$
  3. If $b_T > 0$, balance is due in one *balloon payment*

Returns in practice: debt math

- A debt contract stipulates:
  1. An initial balance b_0
  2. A frequency of payments and total number T of payments (term to maturity)
  3. An interest rate r_t for each period $t = 1,2, \ldots T$
  4. Payments m_t for each period $t = 1,2, \ldots T$
- Debt algebra:
  1. At a given date, interest due is $b_{t-1}\, r_t$
  2. Balance at date t is $b_t = b_{t-1} + b_{t-1} r_t - m_t$
  3. If $b_T > 0$, balance is due in one *balloon payment*

Some language

- Debt contracts whose balance is zero after T periods ($b_T = 0$) are called *fully amortizing*
- Interest rates can be fixed, vary on a fixed schedule, or according to some other market rate
- Amortization can be negative -- balance can grow -- from one period to the next (if $m_t < b_{t-1}r_t$)

Ex: fixed rate coupon bonds

- For all t:
  1. $r_t = r$
  2. $m_t = b_o r$
  3. $b_T = b_0$

Ex: Fixed rate, fixed payment debt

- For all t:
  1. $r_t = r$
  2. $m_t = m$
- Fully amortizing: $b_T = 0$
- What must m be? (Fixed annuity formulae)
- $m = b_0 r / (1 - (1 + r)^{-T})$

The fundamental equation of debt design

- Full amortization means:

$$b_T = 0, \quad \text{or, equivalently,} \qquad b_0 = \sum_{t=1}^{T} \frac{m_t}{(1+r)^t}$$

- More generally:

$$b_0 = \sum_{t=1}^{T} \frac{m_t}{(1+r)^t} + \frac{b_T}{(1+r)^T}$$

- Absent side payments, r is the loan's IRR <u>if all payments are made</u>, i.e. the YTM on the debt contract

Fixed payment example

- $100K$, monthly payments, 10 years, $r = 7\%/12$

1. With full amortization: $m = \$1{,}161.08$
2. With 30K balloon: $m = \$\ 987.76$

Graduated payment example

- A corporation wants to finance 50% of a $\$1M$ investment with some debt
- Cash-flows are expected to increase by 0.5% a month over ten years, so the corporation would like debt payments to match that profile
- A lender is willing to do it with a 7% annual interest rate and a fully amortizing contract
- What is the payment schedule?
- Answer: fundamental equation of debt design

Risk: a deeper dive

- Asset returns are subject to uncertainty
- Let S be the set of possible states of the world
- Roll of a fair dice: $S = \{1,2,3,4,5,6\}$
- An *event* is a subset of S
- Ex: $A = \{2,4,6\}$ is the event that the roll is even
- A *probability distribution* is a function that assigns probabilities to each possible state of the word
- Ex: If dice is fair, $P(s) = 1/6$ for all $s \in \{1,2,3,4,5,6\}$

Random variables

- A random variable X on S attaches a value to each possible state of the world
- Assets (risky strings of cash flows), are random variables
- Ex: X pays $\$1$ if roll of dice is even, nothing otherwise:

$$P(X = 1) = P(s \in \{2,4,6\}) = 0.5$$

Expectations

- The *expected value* of a random variable X is defined as:

$$E(X) = \sum_s P(s)\, X(s)$$

- X pays $\$1$ of roll of dice is even, nothing otherwise:

$$\begin{aligned} E(X) = \; & P(s=1)\times 0 + P(s=2)\times 1 \\ & + P(s=3)\times 0 + P(s=4)\times 1 \\ & + P(s=5)\times 0 + P(s=6)\times 1 = 0.5 \end{aligned}$$

Variances and standard deviations

- $$VAR(X) = \sum_s P(s)\big(X(s) - E(X)\big)^2$$
  $$= E[X - E(X)]2$$

- X pays $\$1$ of roll of dice is even, nothing otherwise:
  $$VAR(X) =$$
  $$P(s = 1)(0 - 0.5)^2 + P(s = 2)(1 - 0.5)^2$$
  $$+ P(s = 3)\,(0 - 0.5)^2 + P(s = 4)(1 - 0.5)^2$$
  $$+ P(s = 5)\,(0 - 0.5)\,2 + P(s = 6)\,(1 - 0.5)2$$

- The *standard deviation* of X is the square root of its variance

Variances and standard deviations

- $VAR(X) \;=\; \sum_s P(s)\big(X(s) - E(X)\big)^2$

  $= E[X - E(X)]2$

- X pays \$1 of roll of dice is even, nothing otherwise:

$$\begin{aligned} VAR(X) = {} & P(s=1)(0-0.5)^2 + P(s=2)(1-0.5)^2 \\ & + P(s=3)\,(0-0.5)^2 + P(s=4)(1-0.5)^2 \\ & + P(s=5)(0-0.5)^2 + P(s=6)\,(1-0.5)2 \\ & = \; 0.25 \end{aligned}$$

- The standard deviation of X is the square root of its variance:

$$\sigma_X = \sqrt{VAR(X)}$$

Risk

- A random variable *X* is *risk-free* if

$$VAR(X) = 0 \Leftrightarrow X(s) = x \text{ for all } s \in S$$

- It is *risky* if $VAR(X) > 0$

- The closest risk we have to risk-free asset in the US (the world?) is a T-bill

Covariance

- We need a notion of how two random variables X and Y are related:

$$COV(X,Y) = \sum_s P(s)\big(X(s) - E(X)\big)\big(Y(s) - E(Y)\big)$$
$$= E[(X - E(X))(Y - E(Y))]$$

- $COV(X,Y) > 0$ means that X tends to be high when Y tends to be high, and vice-versa

- Note 1: if X is risk-free, then $COV(X,Y) = 0$
- Note 2: $COV(X,X) = VAR(X)$
- Note 3: $COV(X,Y) = COV(Y,X)$

Example

- X pays $\$1$ if roll of dice is even, Y pays $\$1$ if roll of dice is 4 or more

- Then $E(X) = E(Y) = 0.5$, and:

$$
\begin{aligned}
COV(X,Y) \; = \; & \\
& P(s=1)(0-0.5)(0-0.5) + P(s=2)(1-0.5)(0-0.5) \\
+ & P(s=3)(0-0.5)(0-0.5) + P(s=4)(1-0.5)(1-0.5) \\
+ & P(s=5)(0-0.5)(1-0.5) + P(s=6)(1-0.5)(1-0.5) \\
& = 1/12
\end{aligned}
$$

Coefficient of correlation

- $\rho_{X,Y} = COV(X,Y)\ /(\sigma_X \sigma_Y)$
- Varies from -1 to 1
- $\rho_{X,Y} = 1$ means that $Y = a\,X + b$, where $a > 0$
- $\rho_{X,Y} = -1$ means that $Y = a\,X + b$, where $a < 0$

Example

- X pays $\$1$ of roll of dice is even, Y pays $\$1$ if roll of dice is 4 or more
- $\rho_{X,Y} = COV(X,Y)\ /(\sigma_X \sigma_Y)\ = \quad \dfrac{\frac{1}{12}}{\sqrt{0.25 \times 0.25}} = \dfrac{1}{3}$

Independence

- X is *independent* of Y if knowing something about X does not change the probability distribution of Y

- If X and Y are independent then $COV(X, Y) = 0$

- If X and Y are *dependent* then knowing X is useful for forecasting Y

- We just need to understand or *model* that dependence in order to exploit it

A very useful expression

- $VAR(aX + bY) =$

$$a^2VAR(X) + b^2VAR(Y) + 2abCOV(X, Y)$$

- In words, when you add/combine two risky assets, the variance of the resulting portfolio depends on how risky each asset is but also on how they co-vary with one-another

Mixing assets

- Let a and b be numbers, and X and Y be the returns on two assets
- Investing a in X and b in Y returns $aX(s) + bY(s)$ in state s
- (a, b), in this context, is called a *portfolio*
- We write $aX + bY$ for the resulting random variable

Big facts

- $E(aX + bY) = aE(X) + bE(Y)$
- $VAR(aX) = a^2 VAR(X) \Leftrightarrow \sigma_{aX} = a\,\sigma_X$
- $VAR(aX + bY) = a^2 VAR(X) + b^2 VAR(X) + 2ab\, COV(X, Y)$
- $VAR(0.5X + 0.5Y) =$
  $0.25 VAR(X) + 0.25 VAR(X) + 0.5\, COV(X, Y)$

Diversification

- Combining risky assets reduces risk unless $\rho_{X,Y} = 1$

- Returns on risky assets that do not covary perfectly tend to offset each-other, at least a little bit

- If they co-vary negatively, diversification is even greater

- If you bet the same amount on both red and black at the roulette, you're taking on virtually no risk

More facts

- $COV(aX + bY, Z) = aCOV(X, Z) + bCOV(Y, Z)$
- And the big monster:

$$\mathrm{VAR}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \mathrm{COV}(X_i, X_j)$$

Towards CAPM

- Consider an investor who can split her wealth across three risky assets:
  1. Bonds ($E(r^B) = 6\%, \sigma(r^B) = 7\%$)
  2. Stocks ($E(r^S) = 15\%, \sigma(r^S) = 10\%$)
  3. Real Estate ($E(r^E) = 6\%, \sigma(r^E) = 7\%$)

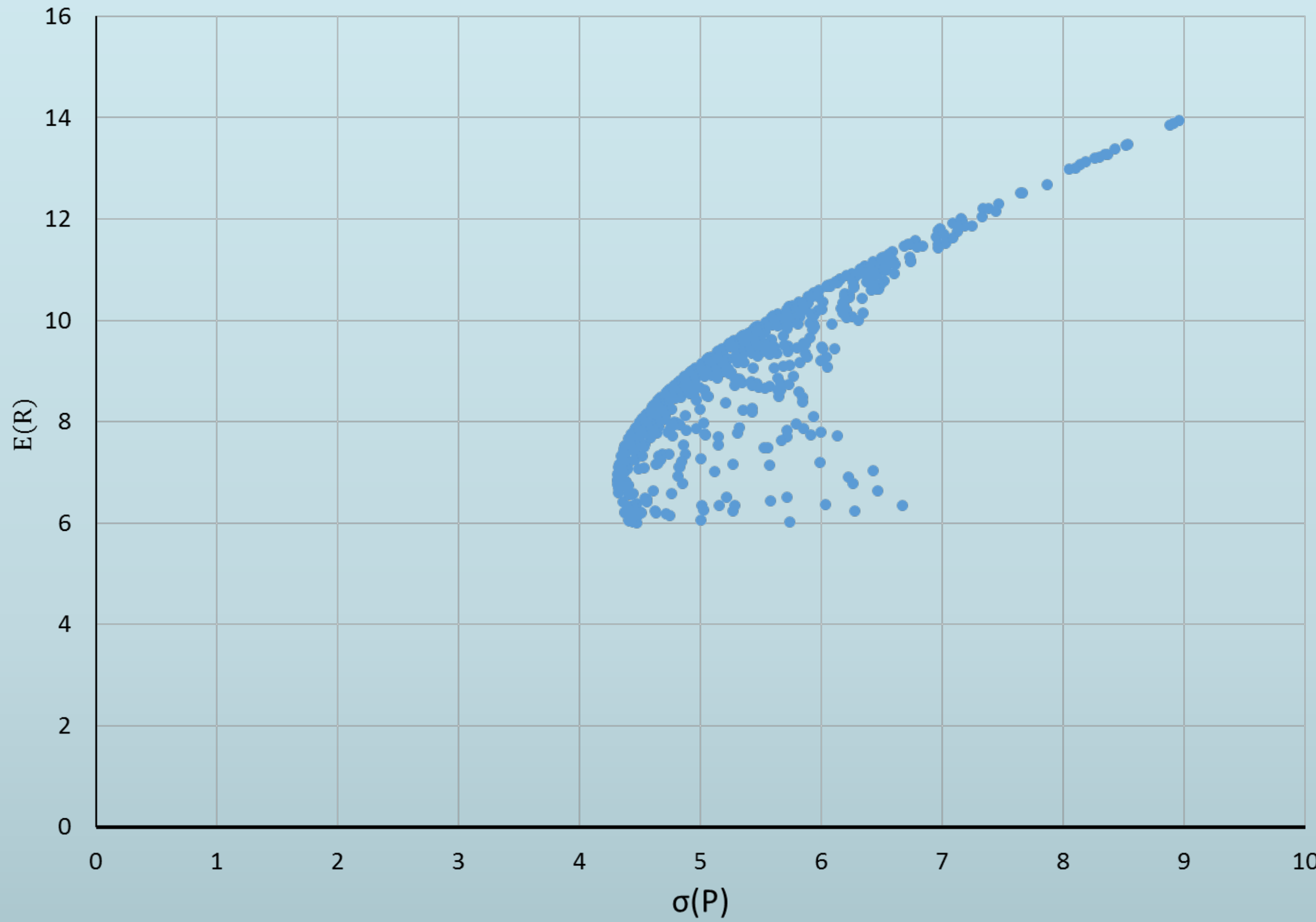
- Covariance matrix (in percentage points) is:

$$\begin{bmatrix} 49 & 10 & -10 \\ 10 & 100 & 10 \\ -10 & 10 & 49 \end{bmatrix}$$

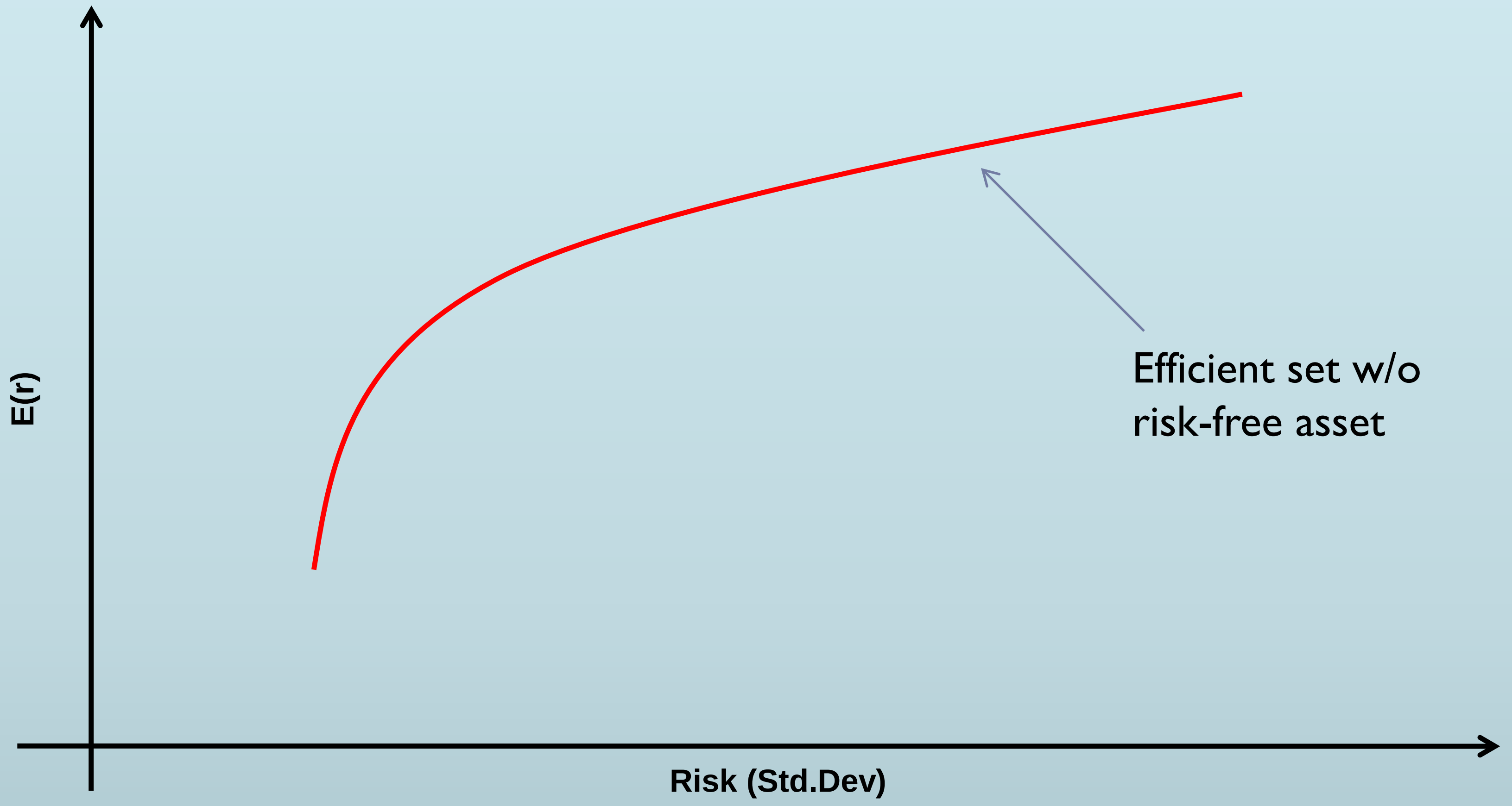
- She likes expected returns, she dislikes variance, she doesn't care about anything else

- That is, she has *mean-variance preferences* and is *risk-averse*

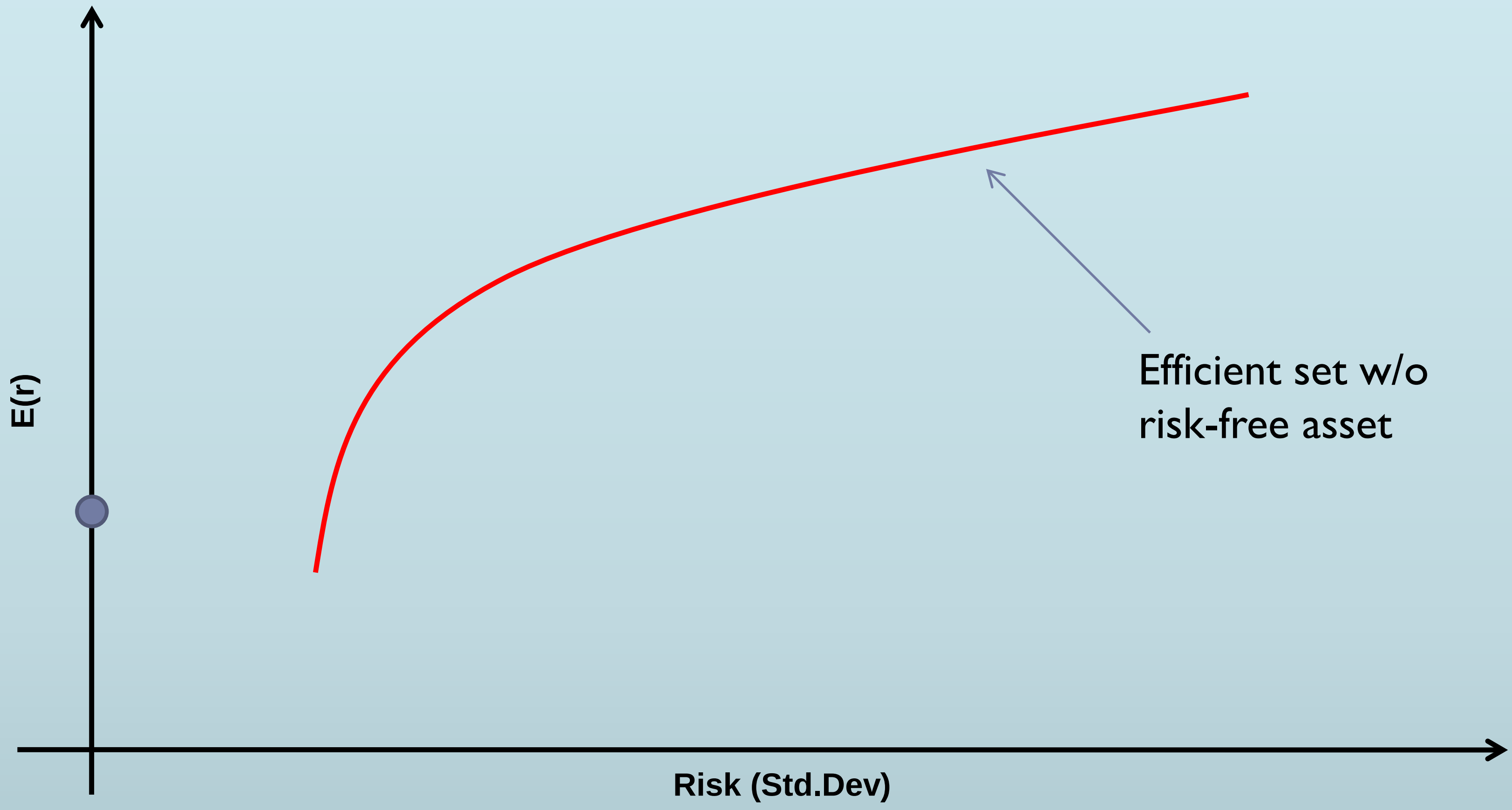
Feasible risk return combinations

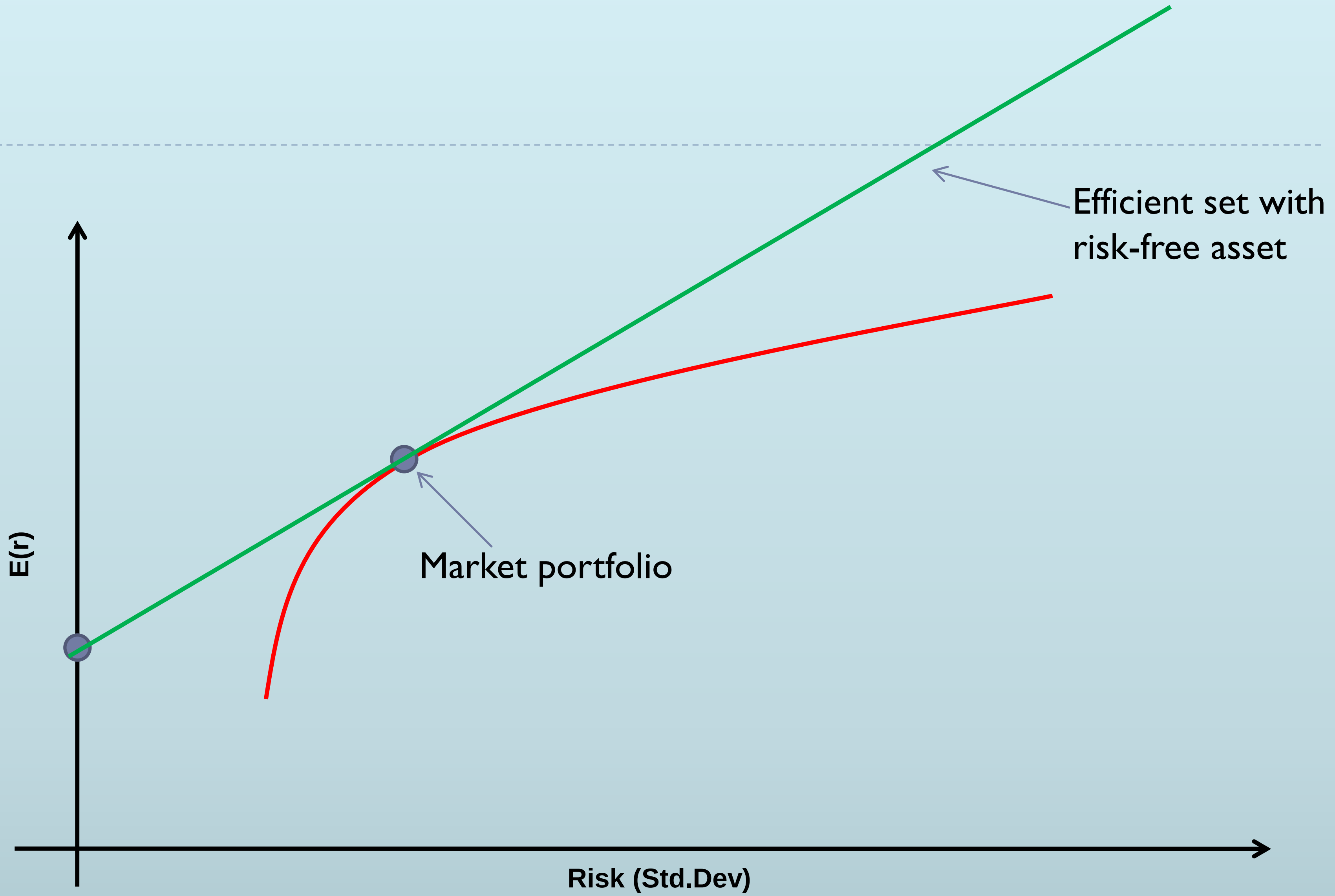


Efficient frontier (no risk-free asset)

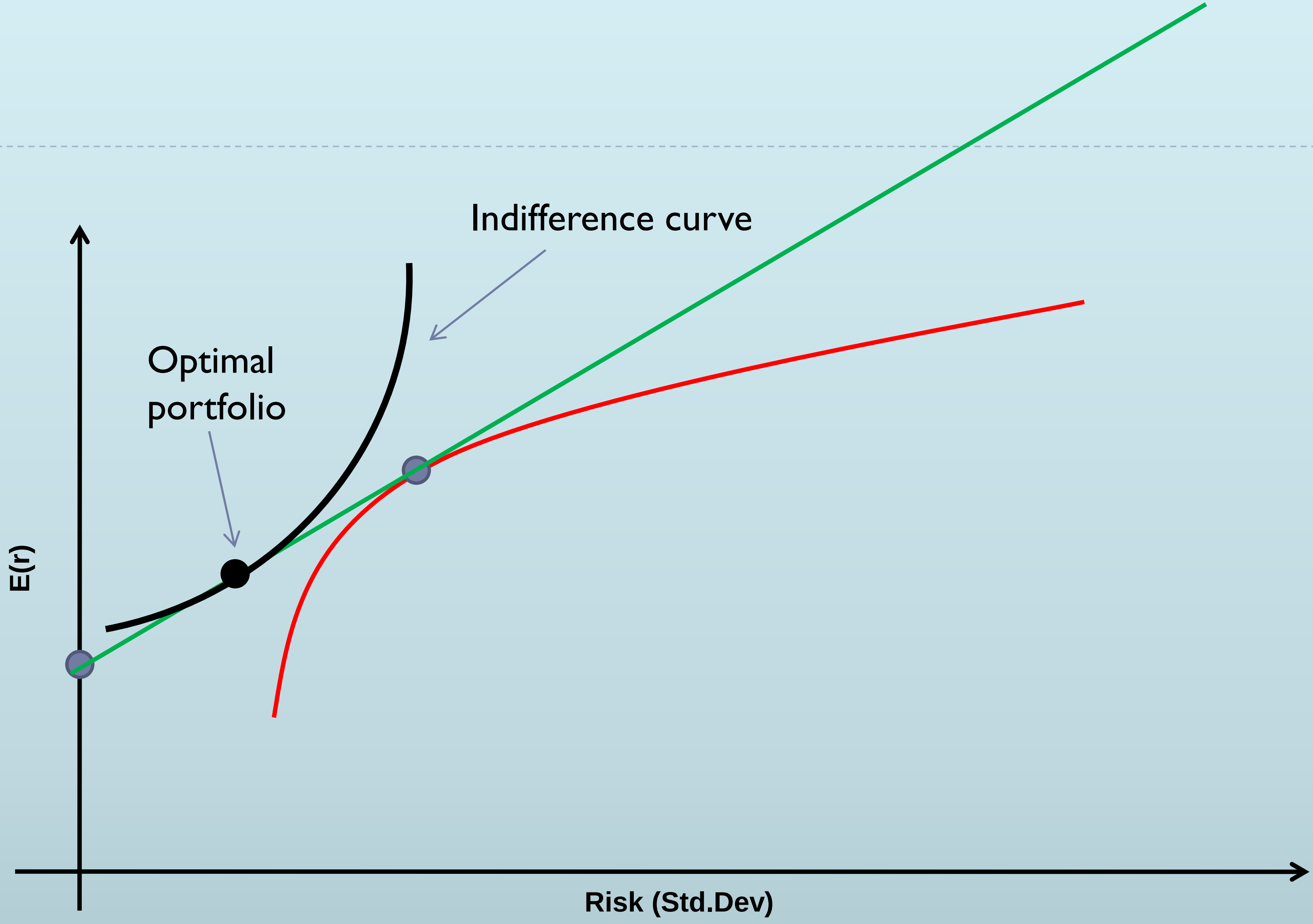


Adding a risk-free asset





Efficient set with
risk-free asset
E(r)
Market portfolio
Risk (Std.Dev)



Indifference curve
Optimal
portfolio
E(r)
Risk (Std.Dev)

Two-portfolio theorem

- With a risk-free asset, efficient set begins at portfolio that puts all wealth in risk-free asset, and touch the risky part of the feasible set in exactly one point
- That point is called the market portfolio
- **Theorem**: In equilibrium, all investors hold a portfolio made of a positive investment in the market portfolio, and a positive or negative investment in the risk-free asset

Market portfolio

- All risky assets have positive weight in it
- The risky-part of all investors portfolios is the same, namely the market portfolios
- It follows that the market portfolio can be computed as the fraction of total risky holdings in a given asset
- Decent practical proxy: capitalization-weighted index, such as the S&P500

Capital Asset Pricing Model (CAPM)

- What should be the average return on asset i in equilibrium? Equivalently, what should be its price?
- Intuitively, riskier assets should command a higher return
- Investors should be compensated for the risk a given asset contributes to their portfolio
- This contribution depends on how it co-varies with all elements of the portfolio, including itself

Capital Asset Pricing Model (CAPM)

- Theorem:

$$E(r_i) = r_f + \beta_i \left[E(r_m) - r_f\right]$$

where:

Capital Asset Pricing Model (CAPM)

- Theorem:

$$E(r_i) = r_f + \beta_i [E(r_m) - r_f]$$

where:

$$\beta_i = \frac{COV(r_i, r_m)}{VAR(r_m)}$$

CAPM

- Investors want to be compensated for a very specific form of risk: the asset's beta
- Return on a given asset is the risk-free rate plus a risk premium
- Risk premium is the product of beta (the *quantity of risk*) and $E(r_m) - r_f$ (the *market price of risk*)

CAPM: a statistical perspective

- What return should I expect from IBM given how I expect the overall market (the S&P500) to perform?
- Assume (heroically) that IBM's beta is time-invariant
- Data: historical returns for IBM and S&P500
- Would any other data be useful? CAPM says no
- In fact, it says that the "best" model for our purposes is a simple *linear regression* model:

$$r_{IBM} - r_f = \alpha + \beta(r_{S\&P} - r_f) + \varepsilon$$

where ε is noise (i.e. mean zero and independent of everything)

CAPM: it fails bigly

- Augment CAPM regression to:

$$r_i - r_f = \alpha_i + \beta_i (r_m - r_f) + \beta_{SML} (SML) + \beta_{HML} (HML) + \varepsilon_i$$

where:

SML = difference between returns on diversified portfolios of small and large capitalization stocks

HML= difference between returns on diversified portfolios of high and low B/M stocks

- The asset/portfolio's empirical intercept (the "alpha") is the return that can't be explained by exposure to FF factors
- Estimates of factor loadings (the beta's) together with forecasts for expected factor values over holding period can be used to calibrate required returns
- Data: Fama-French library