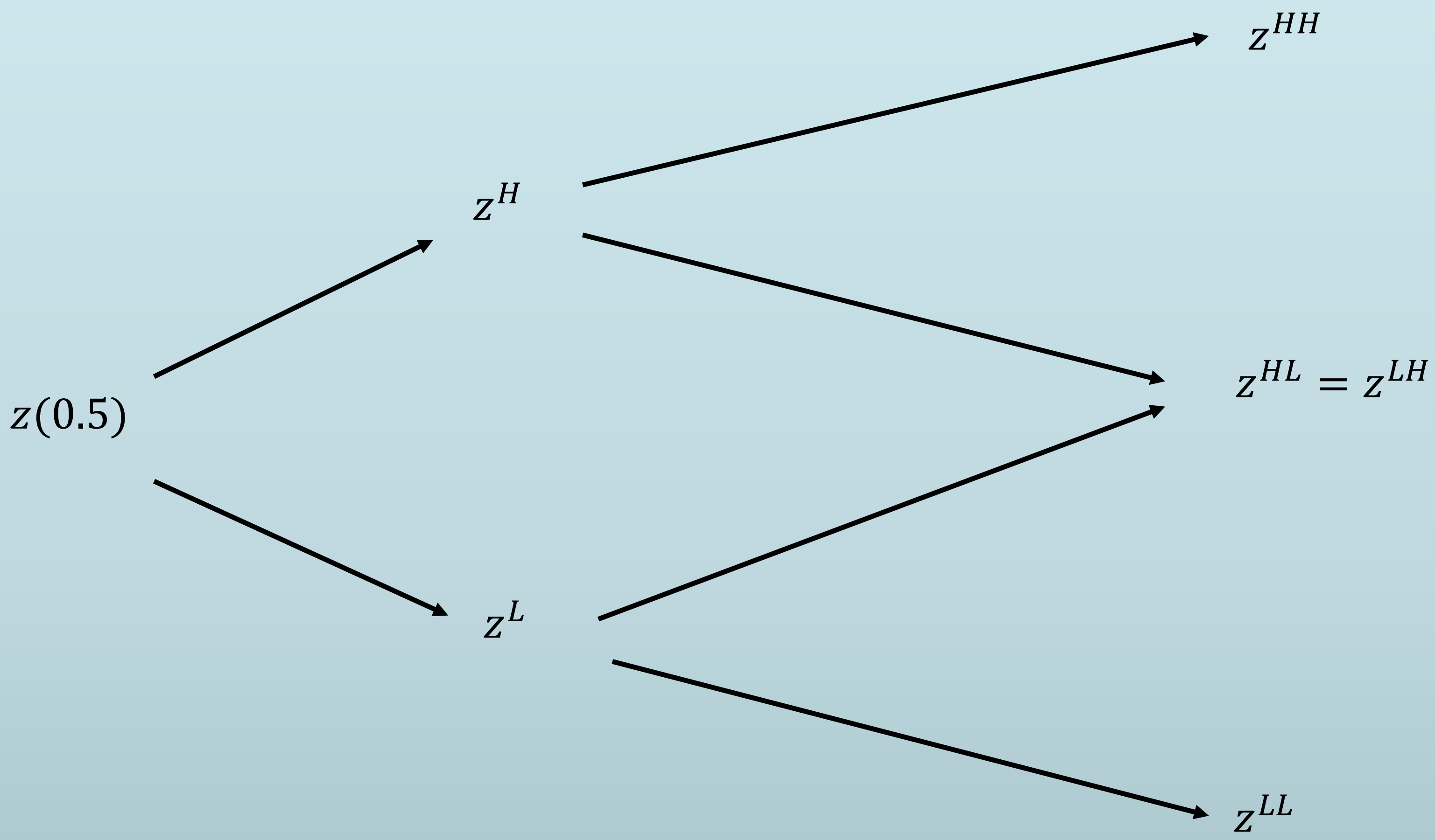
Interest rate models

Fixed income

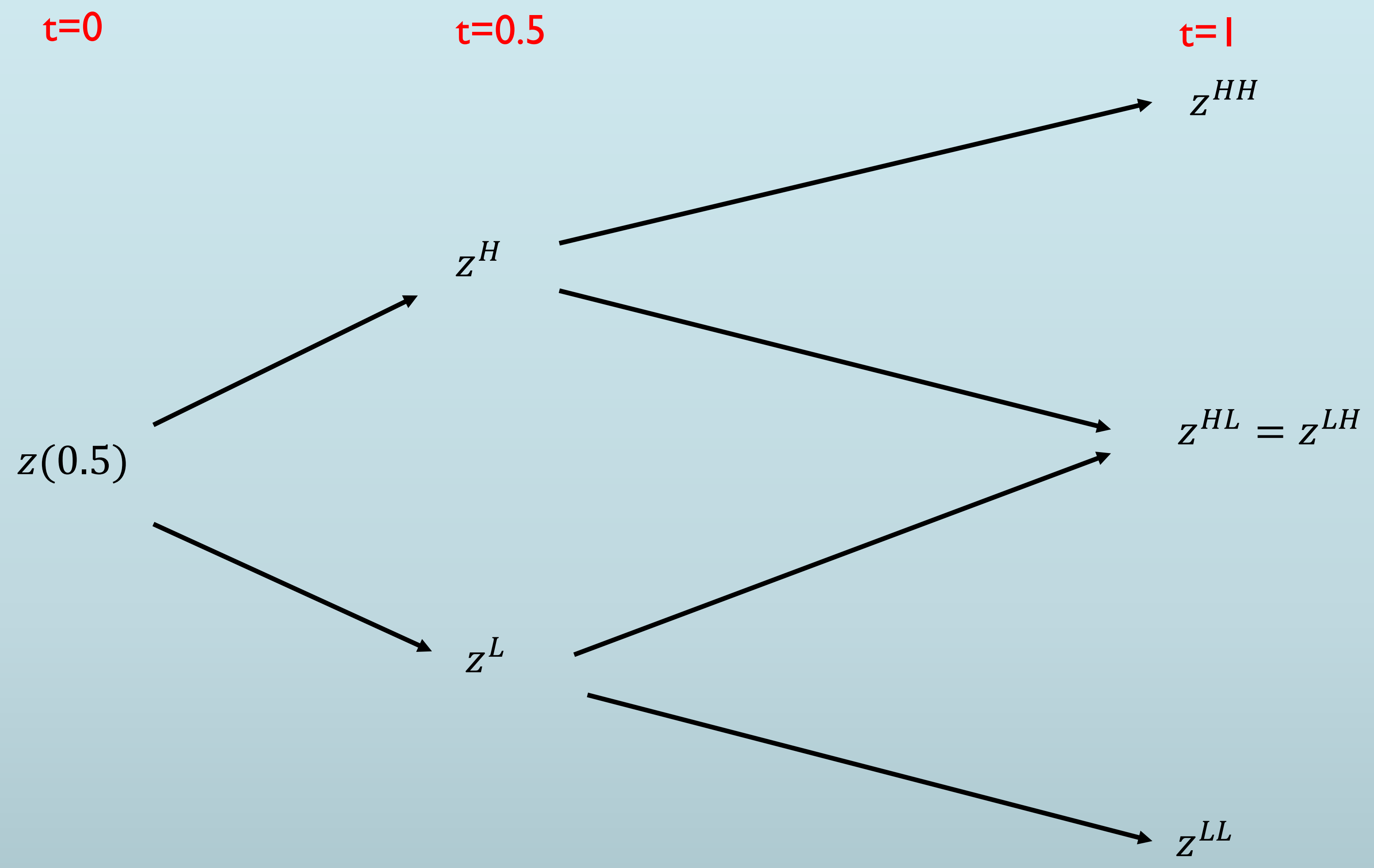
Modeling interest rate uncertainty

- Consider an investment horizon with capital T periods
- The path of T one-period interest rates $(r_1, r_2, \ldots r_T)$ is uncertain, except for the first one
- Assume that the path lives on a binomial tree (rates can go up or down from one period to the next)
- The tree is recombining: value at a given date only depends on total number of ups and downs

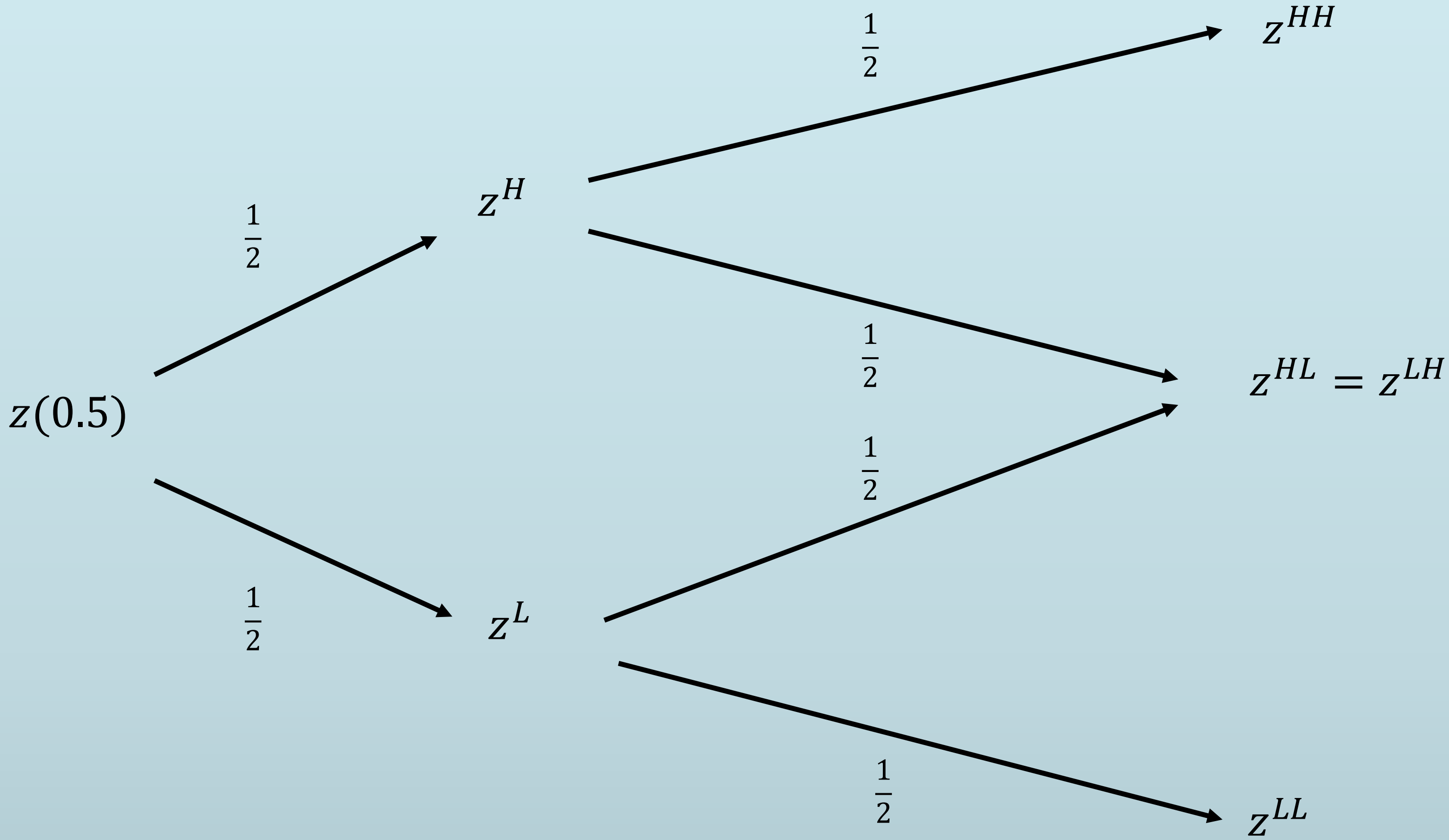
A little tree



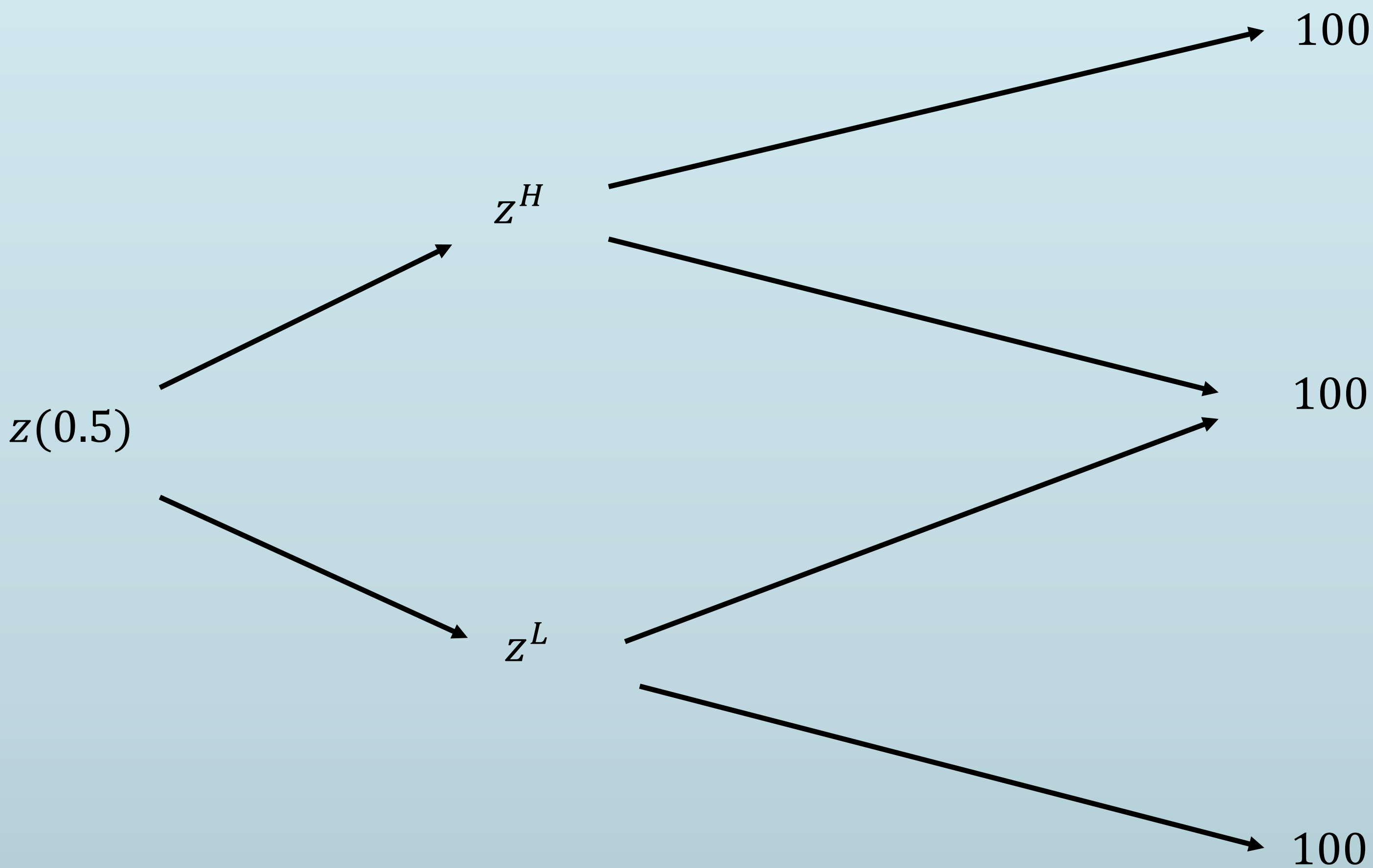
A little tree



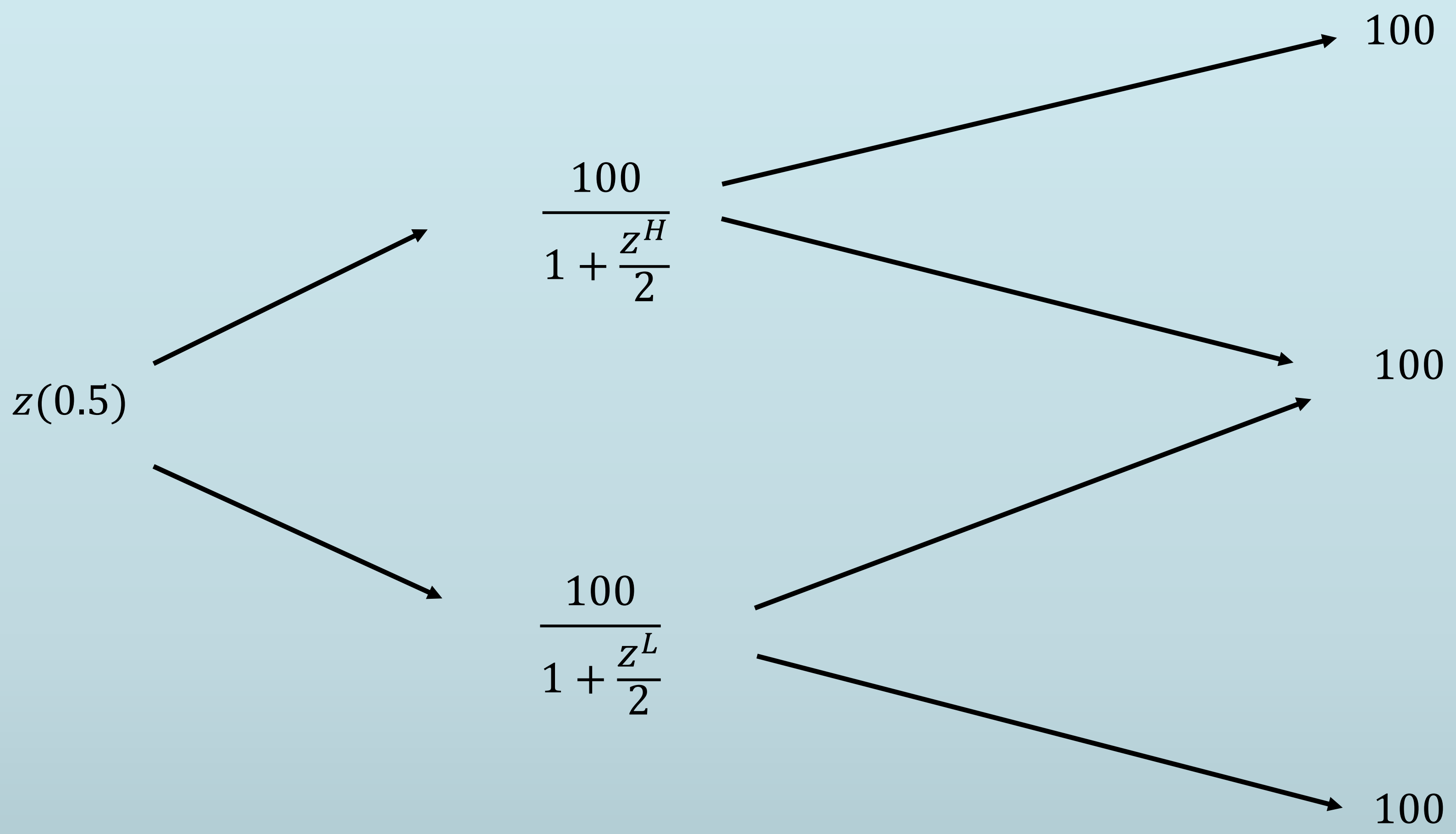
A little tree with probabilities



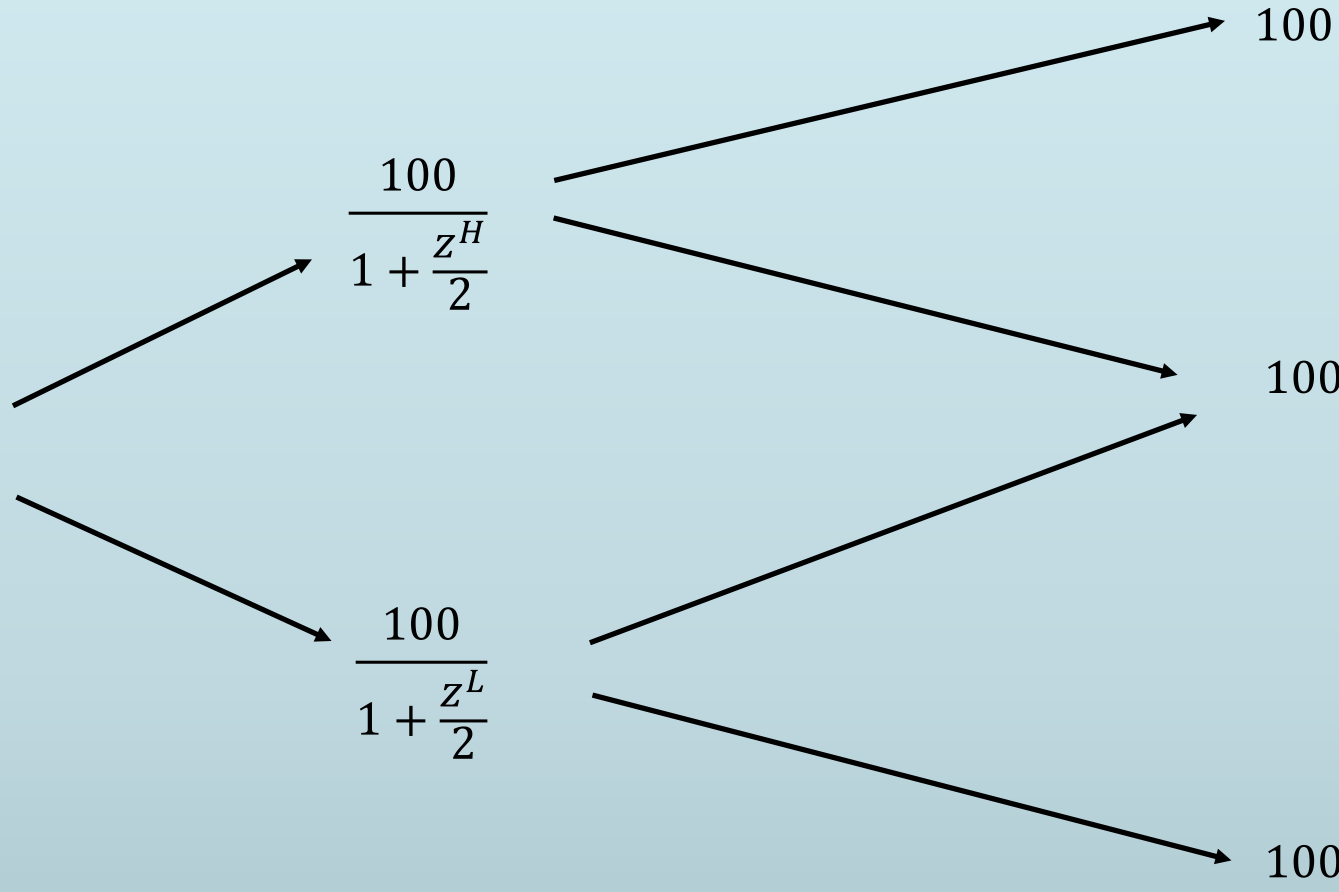
Pricing a 1-year zero



Pricing a 1-year zero



Pricing a 1-year zero



Pricing a 1-year zero

$$\begin{tikzcd}[column sep=large]
 & & 100 \\
 & \frac{100}{1+\frac{z^H}{2}} \arrow[ur] \arrow[dr] & \\
\frac{100}{2\left(1+\frac{z(0.5)}{2}\right)}\left(\frac{1}{1+\frac{z^H}{2}}+\frac{1}{1+\frac{z^L}{2}}\right) \arrow[ur] \arrow[dr] & & 100 \\
 & \frac{100}{1+\frac{z^L}{2}} \arrow[ur] \arrow[dr] & \\
 & & 100
\end{tikzcd}$$

A 6m call on 1-year zero with strike K

$$\max\left(\frac{100}{1+\frac{z^H}{2}} - K, 0\right)$$

$$\frac{1}{2\left(1+\frac{z(0.5)}{2}\right)}\left(\max\left(\frac{100}{1+\frac{z^H}{2}} - K, 0\right) + \max\left(\frac{100}{1+\frac{z^L}{2}} - K, 0\right)\right)$$

$$\max\left(\frac{100}{1+\frac{z^L}{2}} - K, 0\right)$$

Calibration

- We need:
  1. Size of jumps every period
  2. Probability of up or down, under RNP

- Calibrate both to match 1) estimates of interest rate volatility and 2) match the spot-yield curve

- The result is a model that prices treasuries exactly right

- It can and should also price Treasury derivatives trivially

Example: Black-Derman-Troy tree

- Assume that $z(0.5) = 4\ \%$ while $z(1) = 4.5\%$
- Assume that option prices imply that the volatility (std of ln) of rates a year from now is 8%
- Assume symmetric up and down, this implies $z^H = 5.40\%$ and $z^L = 4.60\%$

Two equations, two unknowns

$$\frac{100}{2\left(1+\frac{z(0.5)}{2}\right)}\left(\frac{1}{1+\frac{z^H}{2}}+\frac{1}{1+\frac{z^L}{2}}\right)=\frac{100}{\left(1+\frac{z(1)}{2}\right)^2}$$ (tree prices treasuries)

$$\frac{1}{2}\,ln\left(\frac{z^H}{z^L}\right)=8\%$$ (tree matches rate volatility)

Interest rate caps (or floors)

- Interest caps are a derivative contract on an underlying interest rate rate r_t that pays:

$$\max(0, r_t - \bar{r}) \times A$$

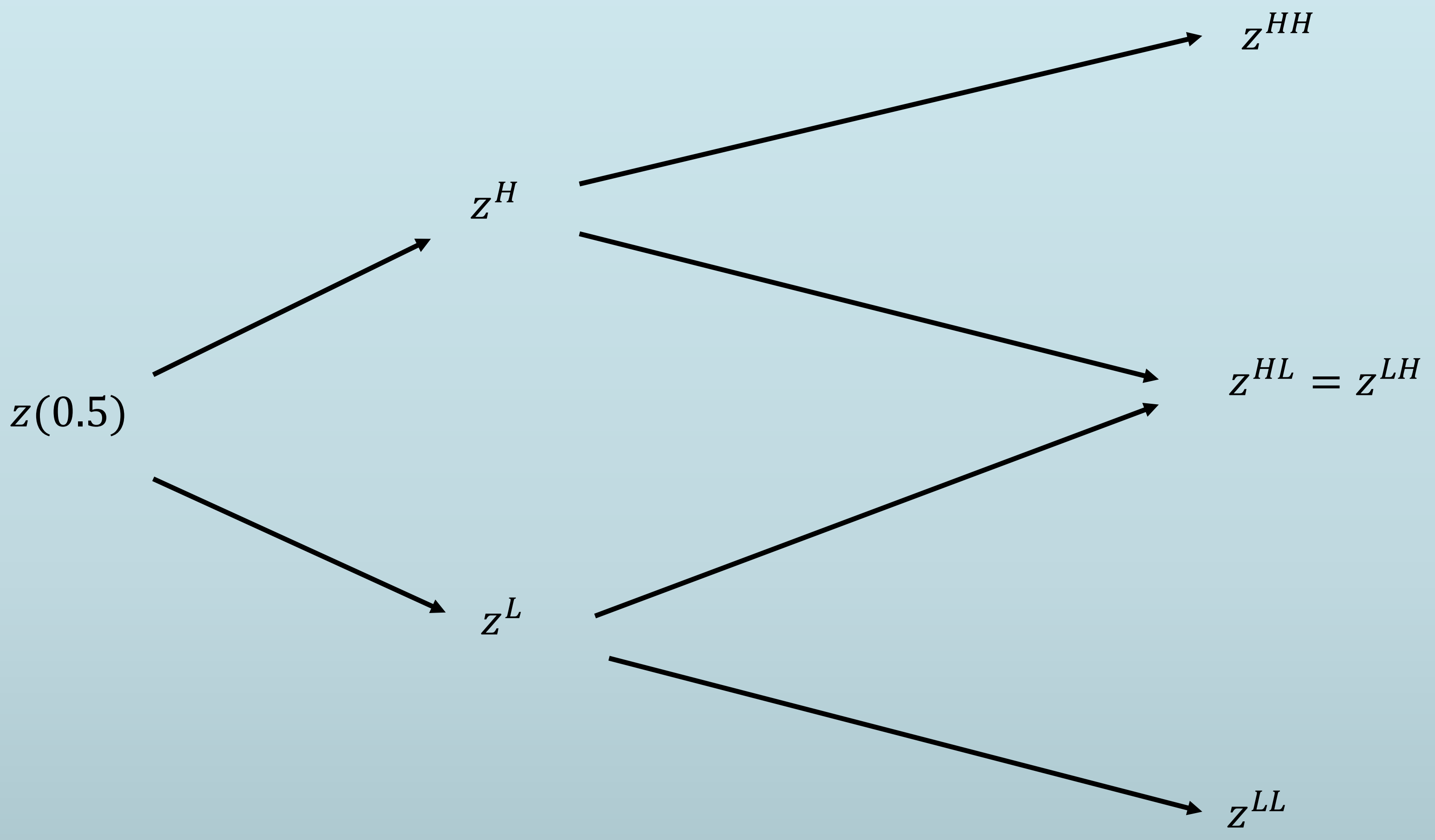
where $\bar{r}$ is a fixed value and A is some notional

- It's a call option so its price implies in the standard option theoretic way a volatility of the underlying rate

How about bonds with options?

- Can a tree, so-built, price bonds that feature default and prepayment risks?
- It should
- But it doesn't
- Enters *OAS*, the measure of our ignorance

A little tree



OAS is a level shift in the tree

$$
\begin{tikzcd}
 & & z^{HH} + OAS \\
 & z^{H} + OAS \arrow[ur] \arrow[dr] & \\
z(0.5) + OAS \arrow[ur] \arrow[dr] & & \begin{array}{c} z^{HL} + OAS \\ = z^{LH} + OAS \end{array} \\
 & z^{L} + \mathrm{OAS} \arrow[ur] \arrow[dr] & \\
 & & z^{LL} + OAS
\end{tikzcd}
$$

OAS implies discount rates

$$D(H) = D(0.5) \times \frac{1}{\left(1 + \frac{z^H}{2} + \frac{OAS}{20000}\right)}$$

$$D(0.5) = \frac{1}{\left(1 + \frac{z(0.5)}{2} + \frac{OAS}{20000}\right)}$$

$$D(L) = D(0.5) \times \frac{1}{\left(1 + \frac{z^L}{2} + \frac{OAS}{20000}\right)}$$

Mortgage pass-through (p33 in FI bank)