Extra notes rendered necessary by shutdown

1 Production math with inventories

Ideally, I want to sell at potential sales (PS). In order to do so, the required inventory at potential sales (RIPS) is

$$RIPS = PS \times IR,$$

where IR, the inventory ratio, is 5% in our case. But of course, except in year 1, I already have some inventory from last year. Call those starting inventory (SI). Production needed to sell at potential (PNP, for short) is

$$PNP = PS + RIPS - SI = PS(1 + IR) - SI.$$

But I am constrained by capacity so units produced (UP) are

$$UP = \min(\text{Capacity}, PNP)$$

Here I am making a strong but convenient assumption, namely that we do not have an incentive or do not have the ability to hoard inventory for later years. In principle, we could produce above PNP to prepare for later years and, depending on how costly it is to store excess inventory, this could be profitable.

With this, we simply note that given units sold (US) end inventory (EI) is given by

$$EI = UP + SI - US$$

while, at the same time (again, under no hoarding) we have

$$EI = US \times IR.$$

Combining the last two expressions gives a simple way to figure sales, namely:

$$US = \frac{UP + SI}{1 + IR}.$$

Now we can take all of this to Excel.

2 COGS math

Under the assumption that I always sell my inventory first (first-in-first-out) my COGS has to be:

$$COGS = SIC + (US - SI) \times Unit Cost$$

where SIC is the book value of my starting inventory, i.e. my starting inventory at cost, while Unit Cost is the production cost per unit. This is the expression we are going to use in our excel work for COGS.

But recall from the previous section that US = UP + SI - EI so replacing US with this in the expression above gives

$$COGS = SIC + UP \times \text{Unit Cost} - EI \times \text{Unit Cost}$$

= $SIC + UPC - EIC$

where UPC is units produced at cost while EIC is ending inventory at cost. This expression should be familiar from your basic accounting courses.

3 Capital gains taxation

Broadly defined, capital gains when an asset is disposed of can be written as:

Capital Gains = NSP – Book Value
= NSP – (Original Basis + All Capex – All Depreciation)
=
$$\underbrace{NSP - (Original Basis + All Capex)}_{(1)} + \underbrace{All Depreciation}_{(2)}$$

The last part (piece 2) gets taxed at the depreciation recapture tax rate while the rest (piece 1) gets taxed at the typically lower ordinary capital gains tax rate.

4 MM with taxes

MM says that, absent taxes and other frictions, $V^U = V^L$. What happens with taxes? FCFF (Note: not unlevered FCFF, I mean bona fide FCFF) when the corporation is taxed are given by:

$$FCFF = (1 - \tau)EBIT + \tau Dr^D.$$

Using the standard "deconstruct the chicken" argument, we can measure market value by pricing those two sub cash-flows one at a time.

The first (i.e. $(1 - \tau)EBIT$) is the cash flows of the unlevered corporation so the proper discount rate for it is $E(r^U)$. The second τDr^D is proportional to debt-cash flows so the right discount rate for that is r^D . It follows that:

$$V^{L} = \frac{(1-\tau)E(EBIT)}{E(r^{U})} + \frac{\tau Dr^{D}}{r^{D}} = V^{U} + \tau D.$$

In words, the value of the levered corporation is the value of the unlevered corporation plus the present value of the debt tax-shield.

5 Leverage mechanics with taxes

We know that $r^U = \frac{(1-\tau)X}{V^U}$ where $(1-\tau)X$, as usual, is unlevered FCFF. On the other hand,

$$r^{E} = \frac{(1-\tau)X - Dr^{D}(1-\tau)}{E} = \frac{(1-\tau)X}{E} - (1-\tau)\frac{D}{E}r^{D}.$$

Mutliply the first term in the last part of the expression by $\frac{V^U}{V^U} = \frac{D + E - \tau D}{V^U}$ to get

$$\begin{aligned} r^{E} &= \frac{(1-\tau)X}{V^{U}} \frac{D+E-\tau D}{E} - (1-\tau) \frac{D}{E} r^{D} \\ &= r^{U} \left(1 + (1-\tau) \frac{D}{E} \right) - (1-\tau) \frac{D}{E} r^{D} \\ &= r^{U} + (1-\tau) \frac{D}{E} \left(r^{U} - r^{D} \right) \end{aligned}$$

6 WACC musings

6.1 WACC invariance, no taxes

$$WACC = \frac{E}{D+E}E(r^{E}) + \frac{D}{D+E}r^{D}$$

$$= \frac{E}{E+D}\left(E(r^{U}) + \frac{D}{E}(E(r^{U}) - r^{D})\right) + \frac{D}{D+E}r^{D}$$

$$= \frac{E}{E+D}E(r^{U}) + \frac{D}{E+D}(E(r^{U}) - r^{D}) + \frac{D}{D+E}r^{D}$$

$$= E(r^{u})$$

6.2 WACC is the right discount rate for unlevered FCFF (the case with no taxes)

This is now trivial. By definition,

$$V^U = \frac{E(EBIT)}{E(r^U)}$$

But since $V^U = V^L$ by the MM theorem and since $WACC = E(r^U)$ by the algebra above, it follows that

$$V^L = \frac{E(EBIT)}{WACC},$$

as needed

6.3 WACC is the right discount rate for unlevered FCFF (even when there are taxes)

That takes a bit more work. We need to show that

$$V^L = V^U + \tau D = \frac{(1-\tau)E(EBIT)}{WACC}.$$

First, a baby step:

Lemma 1. $V^L \times WACC = V^U \times E(r^U)$

Proof.

$$\begin{split} WACC &= \frac{E}{V^{L}} E(r^{E}) + \frac{D}{V^{L}} r^{D} (1 - \tau) \\ &= \frac{E}{V^{L}} \left(E(r^{U}) + (1 - \tau) \frac{D}{E} (E(r^{U}) - r^{D}) \right) + \frac{D}{V^{L}} r^{D} (1 - \tau) \\ &= \frac{E + D - \tau D}{V^{L}} E(r^{U}) \\ &= E(r^{U}) \frac{V^{U}}{V^{L}}, \end{split}$$

which is what we needed to show.

With that tedious algebra done, we are almost there:

$$\frac{(1-\tau)E(EBIT)}{WACC} = \frac{(1-\tau)E(EBIT)}{\frac{V^U \times E(r^U)}{V^L}}$$
$$= \frac{\frac{(1-\tau)E(EBIT)}{E(r^U)}}{\frac{V^U}{V^L}}$$
$$= \frac{V^U}{\frac{V^U}{V^L}}$$
$$= V^L$$

Bottom line, if you discount unlevered FCFF at WACC, you get the value of the levered corporation, as claimed. Something we already checked numerically in chapter 1.

7 Statistical tests of the trade-off theory

The following regression of leverage ratios at quarter t + 1 on the same at quarter t

$$\left(\frac{D}{V}\right)_{t+1} = \gamma \left(\frac{D}{V}\right)^* + (1-\gamma) \left(\frac{D}{V}\right)_t + \epsilon_{t+1}$$

should describe reasonably well the behavior of a corporation that partially adjusts its leverage ratio from where it currently is towards a long-term target $\left(\frac{D}{V}\right)^*$, where

 γ measures the speed of adjustment. The current leverage ratio matters because adjusting one's leverage ratio either by issuing more debt or retiring some debt is costly and takes time. White noise ϵ simply reflects the fact that every one has a plan until they get punched in the mouth (Mike Tyson, 1987.)

This class of equation (a regression of a variable on its most recent value) is known in times series econometrics as an auto-regressive process of order one, or AR(1) for short. Notice that if $\gamma = 0$ the equation degenerates to

$$\left(\frac{D}{V}\right)_{t+1} = \left(\frac{D}{V}\right)_t + \epsilon_{t+1}$$

which is known as a random walk. A corporation whose leverage ratio follows a random walk has no long-term target. Finding evidence of a long-term leverage target, therefore, is rejecting the hypothesis that $\gamma = 0$.

The regression above gives us an estimate of $1 - \gamma$ and so we need to reject the hypothesis that that coefficient is 1. Note that we can't just read that off the standard regression output of excel (except via confidence intervals which, as you know, I find to be dangerous and constantly misused objects, so better to stay away from those, in my opinion) because t-stats and p-values in there refer to the hypothesis that a particular coefficient is zero, not one. But those t-stats simply count how many standard errors (SEs) a coefficient is from zero. We just need, instead, to count how many SEs the coefficient on $\left(\frac{D}{V}\right)_t$ is from 1. The t-stat we want, then, is

$$\frac{1 - \text{Coefficient on } \left(\frac{D}{V}\right)_t}{SE}.$$

Unfortunately, that t statistic, although well defined, does not follow a standard tdistribution. That's because under the hypothesis that $\gamma = 0$, the leverage ratio time series is not stationary and ergodic. Delving into this further is beyond the scope of this class but suffice it to say that different critical values need to be used in this context (for details, google DickeyFuller test). Details aside, one can reject the hypothesis that $\gamma = 0$ in favor of the alternative hypothesis that $\gamma > 0$ as long as the above t-stat is big enough in absolute value. A critical value of 3 is safe for the specific model we are looking at here as long as you have at least 25 periods of data.

8 Distress

Consider a corporation for which distress is a positive probability event. Specifically, to take the simplest possible example, take an untaxed corporation that is going to live exactly one more period. Absent any new investment, it will generate either \$80M or \$100M, each with equal probability. Existing debt holders are owed \$90M in the final period so, as is, the corporation may need to default on its obligations.

The corporation can add a new project at a cost of \$10*M*. The project would generate net cash-flows *X* in the final period with probability one. It must be financed with equity, however, and investors into this new project require an expected return of 5%. As a stand alone project, this would be a positive NPV project as long as $X \ge $10.5M$.

But the project cannot be separated from the corporation. And existing debt-holders have priority over any remaining share-holders next period. How high must be X to justify investing into the new project? Since debt holder will "tax" 10M out of the project if the corporation is only able to generate 80M in other cash-flows in the final period, we need:

$$\frac{1}{2}X + \frac{1}{2}(X - 10) \ge 10(1 + 5\%)$$

which requires $X \ge 15.5$.

This is called a **debt-overhang problem.** The risk of default may cause the corporation to pass on positive NPV projects.

Another possible consequence of the proximity of default is it may cause the owners of a corporation or its managers to take excessive risks. To see this, imagine that the corporation owes 100M in debt in the final period. Then share-holders are going to be wiped out absent any new investment. So assume those shareholders can enter at no cost into a risky project that pays 10M in the good state and -1,000,000,000,000,000M in the bad state (this is enlarged to show texture.) While this is a negative NPV project by any measure for anybody but incumbent shareholders, it is worth taking on for those incumbents. They earn zero regardless in the bad state due to limited liability, but stand to make some money in the good state. This phenomenon, for obvious reasons, is referred to as **gambling for resurrection**.

9 The market value of debt

Consider a corporation with a coupon (interest only) piece of debt on its books with a fixed interest rate of r, T periods to maturity, and an outstanding balance of b. Market rates on this sort of instrument are now r'. The market value of the instrument is:

$$\begin{split} \sum_{t=1}^{T} \frac{br}{(1+r')^{t}} + \frac{b}{(1+r')^{T}} &= br \frac{\frac{1}{1+r'} - \left(\frac{1}{1+r'}\right)^{T+1}}{1 - \frac{1}{1+r'}} + \frac{b}{(1+r')^{T}} \\ &= br \frac{1 - \left(\frac{1}{1+r'}\right)^{T}}{r'} + \frac{b}{(1+r')^{T}} \end{split}$$

as implied by standard geometric sum algebra. Note that, naturally, if r = r' then book value and market value match (check that you see it.) To compute the market value of a corporation's debt then, we need to apply this sort of math to each and every one of the corporation's outstanding fixed rate instruments.

Or we can use a short-cut, a short-cut often rendered necessary by lack of data in practice. The short-cut involves estimating the corporations weighted average contract rate (WACR), its weighted average market rate (WAMR), and its weighted average remaining maturity (WARM). For a corporation with outstanding debt of face value FV_t , contract rate r_t and market rate r'_t for maturities t = 1, 2, ..., T, in order,

$$WACR = \frac{\sum_{t=1}^{T} FV_t r_t}{\sum_{t=1}^{T} FV_t}$$
$$WAMR = \frac{\sum_{t=1}^{T} FV_t r'_t}{\sum_{t=1}^{T} FV_t}$$
$$WARM = \frac{\sum_{t=1}^{T} FV_t}{\sum_{t=1}^{T} FV_t}$$

Then you close your eyes, pretend that the corporation has one piece of debt of size $b = \sum_{t=1}^{T} FV_t$, of maturity equal to WARM, of coupon rate (r) equal to WACR, and pretend that bond's market interest rate (r') would be WAMR. In that ideal world, the one-bond equation above is all you need.

10 Refinancing math, given tax implications

Since interest is associated with a tax shield, our refinancing calculations of chapter 1 were incomplete (in other words, only exactly right for a tax-free corporation.) We need to substract the present value of the lost tax shield from the NPV of refinancing but, thankfully, we now know how to do that. Measure the NPV of refi as if the corporation were untaxed but then substract the PV of the lost debt-tax shield. For instruments that are not interest-only this requires a detailed computation of interest payments until the bond is retired.

11 The pecking order view

The following example is drawn directly of Myers and Majluf (1984). It shows that incumbent shareholders and/or managers representing them are less likely to raise new equity when they have superior information that suggests that the firm is currently overvalued. Raising new equity thus causes external investors to revise their expectations down and the firm value to fall.

Consider a financial economy in which – purely for simplicity –all investors are riskneutral and do not discount the future. Put another way, they are willing to pay the expected payoff for any investment project, however distant the payoff. An unlevered corporation has assets in place which, next period, will be worth either 150 in the good state or 50 in the bad state. Both states are equally probable. The true value of those assets will be revealed once the state is realized.

The corporation also has the option to invest in a new project at a cost of 100. The new project will pay 110 next period in the bad state and 120 in the good state. So, in particular, it is a positive NPV project and, in well functioning markets with full information, would be undertaken for sure.

The new project has to be financed with new equity. This, unlike all our simplications so far, is a critical assumption. Existing (old) shareholders have superior information: they learn if the state is going to be good or bad before investing in the new project.

Assume – towards reaching a contradiction – that old shareholders choose to invest no matter the information they have about the new state. In that case, outside investors (aka the market) learns nothing from the investment decision so the market value of

the firm following the investment has to be

$$\frac{1}{2}(150 + 120) + \frac{1}{2}(50 + 110) = 215.$$

Since 100 in new equity has to be raised, the remaining share of the firm in the hands of old shareholders is $\frac{215-100}{215}$.

But this investing-no-matter-what strategy cannot be part of an equilibrium. Old shareholders who learn before the equity round that the state is good would earn 150 if they don't raise new equity and and simply wait. If they raise equity and undertake the project they only earn $\frac{115}{215} \times 270 < 150$.

Basically, old investors who have positive information have to give up too much equity under neutral market expectations. The only equilibrium in this world, one checks with a bit more algebra, is one where old investors who get bad news raise new equity and invest while those who get good news do not. So raising new equity signal bad news and that equity is currently overvalued.

12 Dividend math

Assume that investors are eligible for dividend d provided they appear on the company's ledger at some record date. At the close of the date before ex-dividend date t, a gap of size d appears between the unadjusted price P_t and the adjusted price \tilde{P}_t so that

$$P_t = \tilde{P}_t + d.$$

After that most recent dividend, the two series match.

According to the adjusted price series, the return for an investor who buys the stock at the close of date t and sells at the close of date t + 1 is

$$\tilde{r}_t = \frac{\tilde{P}_{t+1} - \tilde{P}_t}{\tilde{P}_t}$$
$$= \frac{P_{t+1} + d - P_t}{P_t - d}$$
$$= \frac{P_{t+1}}{P_t - d} - 1$$

According to the raw series, assuming we invest the dividend at a gross return of x on the ex-dividend day, the return to the same buy-and-sell strategy is

$$\begin{aligned} r_t &= \frac{P_{t+1} + dx - P_t}{P_t} \\ &= \frac{P_{t+1}}{P_t} + \frac{dx}{P_t} - 1 \end{aligned}$$

Some algebra with no redeeming qualities (but you should do it) shows that the two returns are equal if and only if

$$x = \frac{P_{t+1}}{P_t - d}.$$

So the adjusted series gives us the right return under the assumption that dividends invested on the ex-dividend date earn the return implicit in the adjusted series. Mouthful of a sentence.

13 Dividends financeed by stock issue

Consider a firm whose market value is MV, which currently has N shares outstanding (each currently worth $\frac{MV}{N}$), whose debt has market value D, and which wishes to give a dividend d to existing shareholders by issuing x new shares. The number of new shares must solve:

$$x\frac{MV-D}{N+x} = Nd$$
$$\iff x = \frac{N^2d}{MV-D-Nd}$$

14 The option to abandon

Consider an investment project whose continuation value at date 1 is either 100 or 50. It can be activated in two distinct location. In location 1, there is no exit strategy and you are stuck with the project. In location 2, the project can be scrapped for 60 instead of begin continued and upon discovering the realized continuation value. In other words, project 2 is just like project one except that it comes with an option to

scrap if bad news arrive. It follows that:

$$NPV$$
(project 2) = NPV (project 1) + value of option to scrap.

Now assume that the market value of the location 1 project is 70. And assume that the risk-free rate is $r_F = 5\%$. What is the value of the option to scrap? What is the value of location 2 project?

We will use a replication argument to answer those questions, an argument in the tradition of option pricing algebra. The option to scrap has payoff 10M if the owner discovers that the project is bad and 0 otherwise. Let us build an investment portfolio with exactly the same payoff in each state but whose cost we know. Invest an amount Ma (this one is a dollar amount) in the risk-free asset and buy a fraction x (this one is a percentage, and may be negative if you want to short-sell the project) of the project in location 1. The cost of this portfolio today is a + 70x at date 0.

At date 1, it pays $a(1 + r_F) + 100x$ of the payoff is high and $a(1 + r_F) + 50x$ if the payoff is low. Replication thus requires:

$$a(1+r_F) + 100x = 0$$

$$a(1+r_F) + 50x = 10$$

A moment of algebra gives you $x = -\frac{1}{5}$ (so that you want to short-sell 20% of the project in location 1) while $a = \frac{20}{1+5\%} \approx \$M19.05$. The cost of this strategy is

$$a + 70x \approx \$M5.05.$$

That's the value of the option to scrap. The total value of the project in location 2 is simply that value plus the value of the project in location 1, so roughly M75.05.

15 The option to delay refinancing

We apply the exact same logic to the option to delay refinancing. Consider an untaxed corporation which has the option to prepay (call) a bond with 5 years to maturity, b = \$M100 in remaining principal, a 10% yearly rate r_{old} , fixed and monthly payments. It can replace it with a 5-year bond with the same payment structure but a $r_{new} = 9\%$ yearly rate. Prepayment penalties are 2% of the outstanding principal. "Tomorrow,"

rates will be either $r_L = 8.5\%$ or 9.5%. The risk-free rate between today and tomorrow is $r_F = 0.005\%$. For simplicity, assume that the option to refinance will expire after tomorrow, say because the corporation somehow knows that rates will not fall any further after that.

Recall from chapter 1 that, ignoring the option to delay, refinancing has current value $PV(m, r_{new}) - (b+c)$ where m is the current payment and here PV stands for present value of all remaining payments. By the same logic, delaying pays

$$y_L \equiv \max \left\{ PV(m, r_L) - (b+c), 0 \right\}$$

if rates are r_L tomorrow while it pays

$$y_H = \max \{ PV(m, r_H) - (b + c), 0 \}$$

if rates are r_H tomorrow. Now we follow the exact same approach as in the previous section. Invest and amount a at the risk-free rate today and buy (or sell if negative) fraction x of the bond today. The cost of this strategy today is $a + xPV(m, r_{new})$. To replicate, the strategy must solves:

$$a(1+r_F) + xPV(m,r_H) = y_H$$

$$a(1+r_F) + xPV(m,r_L) = y_L$$

The exam same algebra as above gives

$$x = \frac{y_H - y_L}{PV(m, r_H) - PV(m, r_L)}$$

while

$$a = \frac{y_H - xPV(m, r_H)}{1 + r_F}$$

Plugging these values into $a + xPV(m, r_{new})$ gives the value of the option to wait to refi.

Plugging in the specific numerical values above (we will use excel for that purpose like always) shows that the NPV of refinancing abstracting from the option value is M354.18 or so. The replication portfolio entails a = -M65,968.88 and x = 65.21%so that you are borrowing to take a positive position in the bond today. The cost of this portfolio is M777.19 and that is the value of the option to refinance. Since it is higher than the NPV of refinancing before option considerations, you should wait. More surprisingly perhaps, you will find that this is true for just about any parameter configurations (those consistent with no arbitrage, more on that subtle point in your FIN 330 course.) This is a manifestation of the general theorem that it is never optimal to exercise and American call option on an asset that does not pay dividends. Here, the role of dividends is played by the bond payment so what we have learned is something fairly intuitive: it does not make sense to exercise the option to refi in between payments. The only time it makes sense to refinance is just before a payment is due.