Extra notes rendered necessary by shutdown

1 Production math with inventories

Ideally, I want to sell at potential sales (PS). In order to do so, the required inventory at potential sales ($RIPS$) is

$$RIPS = PS \times IR,$$

where IR, the inventory ratio, is 5% in our case. But of course, except in year 1, I already have some inventory from last year. Call those starting inventory (SI). Production needed to sell at potential (PNP, for short) is

$$PNP = PS + RIPS - SI = PS(1 + IR) - SI.$$

But I am constrained by capacity so units produced (UP) are

$$UP = \min(\text{Capacity}, PNP)$$

Here I am making a strong but convenient assumption, namely that **we do not have an incentive or do not have the ability to hoard inventory for later years.** In principle, we could produce above PNP to prepare for later years and, depending on how costly it is to store excess inventory, this could be profitable.

With this, we simply note that given units sold (US) end inventory (EI) is given by

$$EI = UP + SI - US$$

while, at the same time (again, under no hoarding) we have

$$EI = US \times IR.$$

Combining the last two expressions gives a simple way to figure sales, namely:

$$US = \frac{UP + SI}{1 + IR}.$$

Now we can take all of this to Excel.

2 COGS math

Under the assumption that I always sell my inventory first (first-in-first-out) my $COGS$ has to be:

$$COGS = SIC + (US - SI) \times \text{Unit Cost}$$

where SIC is the book value of my starting inventory, i.e. my starting inventory at cost, while Unit Cost is the production cost per unit. This is the expression we are going to use in our excel work for $COGS$.

But recall from the previous section that $US = UP + SI - EI$ so replacing US with this in the expression above gives

$$\begin{aligned} COGS &= SIC + UP \times \text{Unit Cost} - EI \times \text{Unit Cost} \\ &= SIC + UPC - EIC \end{aligned}$$

where UPC is units produced at cost while EIC is ending inventory at cost. This expression should be familiar from your basic accounting courses.

3 Capital gains taxation

Broadly defined, capital gains when an asset is disposed of can be written as:

$$\begin{aligned} \text{Capital Gains} &= \text{NSP} - \text{Book Value} \\ &= \text{NSP} - (\text{Original Basis} + \text{All Capex} - \text{All Depreciation}) \\ &= \underbrace{\text{NSP} - (\text{Original Basis} + \text{All Capex})}_{(1)} + \underbrace{\text{All Depreciation}}_{(2)} \end{aligned}$$

The last part (piece 2) gets taxed at the depreciation recapture tax rate while the rest (piece 1) gets taxed at the typically lower ordinary capital gains tax rate.

4 MM with taxes

MM says that, absent taxes and other frictions, $V^U = V^L$. What happens with taxes? FCFF (Note: not unlevered FCFF, I mean bona fide FCFF) when the corporation is taxed are given by:

$$FCFF = (1-\tau)EBIT + \tau D r^D.$$

Using the standard "deconstruct the chicken" argument, we can measure market value by pricing those two sub cash-flows one at a time.

The first (i.e. $(1-\tau)EBIT$) is the cash flows of the unlevered corporation so the proper discount rate for it is $E(r^U)$. The second $\tau D r^D$ is proportional to debt-cash flows so the right discount rate for that is r^D. It follows that:

$$V^L = \frac{(1-\tau)E(EBIT)}{E(r^U)} + \frac{\tau D r^D}{r^D} = V^U + \tau D.$$

In words, the value of the levered corporation is the value of the unlevered corporation plus the present value of the debt tax-shield.

5 Leverage mechanics with taxes

We know that $r^U = \frac{(1-\tau)X}{V^U}$ where $(1-\tau)X$, as usual, is unlevered FCFF. On the other hand,

$$r^E = \frac{(1-\tau)X - Dr^D(1-\tau)}{E} = \frac{(1-\tau)X}{E} - (1-\tau)\frac{D}{E}r^D.$$

Mutliply the first term in the last part of the expression by $\frac{V^U}{V^U} = \frac{D+E-\tau D}{V^U}$ to get

$$\begin{aligned} r^E &= \frac{(1-\tau)X}{V^U}\frac{D+E-\tau D}{E} - (1-\tau)\frac{D}{E}r^D \\ &= r^U\left(1+(1-\tau)\frac{D}{E}\right) - (1-\tau)\frac{D}{E}r^D \\ &= r^U + (1-\tau)\frac{D}{E}\left(r^U - r^D\right) \end{aligned}$$

6 WACC musings

6.1 WACC invariance, no taxes

$$\begin{aligned}
WACC &= \frac{E}{D+E}E(r^E) + \frac{D}{D+E}r^D \\
&= \frac{E}{E+D}\left(E(r^U) + \frac{D}{E}(E(r^U) - r^D)\right) + \frac{D}{D+E}r^D \\
&= \frac{E}{E+D}E(r^U) + \frac{D}{E+D}(E(r^U) - r^D) + \frac{D}{D+E}r^D \\
&= E(r^u)
\end{aligned}$$

6.2 WACC is the right discount rate for unlevered FCFF (the case with no taxes)

This is now trivial. By definition,

$$V^U = \frac{E(EBIT)}{E(r^U)}$$

But since $V^U = V^L$ by the MM theorem and since $WACC = E(r^U)$ by the algebra above, it follows that

$$V^L = \frac{E(EBIT)}{WACC},$$

as needed

6.3 WACC is the right discount rate for unlevered FCFF (even when there are taxes)

That takes a bit more work. We need to show that

$$V^L = V^U + \tau D = \frac{(1-\tau)E(EBIT)}{WACC}.$$

First, a baby step:

Lemma 1. $V^L \times WACC = V^U \times E(r^U)$

Proof.

$$
\begin{aligned}
WACC &= \frac{E}{V^L}E(r^E) + \frac{D}{V^L}r^D(1-\tau) \\
&= \frac{E}{V^L}\left(E(r^U) + (1-\tau)\frac{D}{E}(E(r^U) - r^D)\right) + \frac{D}{V^L}r^D(1-\tau) \\
&= \frac{E + D - \tau D}{V^L}E(r^U) \\
&= E(r^U)\frac{V^U}{V^L},
\end{aligned}
$$

which is what we needed to show. □

With that tedious algebra done, we are almost there:

$$
\begin{aligned}
\frac{(1-\tau)E(EBIT)}{WACC} &= \frac{(1-\tau)E(EBIT)}{\frac{V^U \times E(r^U)}{V^L}} \\
&= \frac{\frac{(1-\tau)E(EBIT)}{E(r^U)}}{\frac{V^U}{V^L}} \\
&= \frac{V^U}{\frac{V^U}{V^L}} \\
&= V^L
\end{aligned}
$$

Bottom line, if you discount unlevered FCFF at WACC, you get the value of the levered corporation, as claimed. Something we already checked numerically in chapter 1.